Employment Uncertainty and Wage Contracts in Frictional Labor Markets *

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Abstract

Two essential aspects of many employment relationships are, (1) that they are meant to last a long time, and (2) that the participation of the worker and the firm in the match cannot be enforced during the relationship. I construct a model which takes the bilateral uncertainty of participation seriously, and characterize the optimal contract. In a frictional labor market, risk-neutral workers randomly receive opportunities to leave the firm. At the same time they are uncertain about their productivity in the match, and therefore about the (future) participation of the firm. Thus, in equilibrium, each party’s participation decision depends on the expected participation decisions on the other side. In the model, firms post wage-tenure contracts, trading off instantaneous profit with the need to retain its workers, especially the workers in high-productivity matches. In the resulting contract, wages exhibit gradual growth over the course of the employment relationship, driven by (and responsive to) precisely the bilateral nature of uncertainty. This effect of uncertainty is in stark contrast to the previous literature. The model can be readily extended to study the market equilibrium. It can thus speak to important features of the data, such as wage growth with tenure, the decline of turnover and dismissals with the length of an employment relationship, and the link between the extent of bilateral participation uncertainty and aggregate labor market outcomes.

*Job Market Paper. The most up-to-date version of this paper can be downloaded from my webpage www.econ.upenn.edu/~lpv/lpvpaper.pdf. For many discussions, and a lot of inspiration, I thank Ken Burdett, Randy Wright, Iourii Manovskii and Guido Menzio. I also would like to thank Jan Eeckhout, Manolis Galenianos, Philipp Kircher, Ben Lester, Auroco de Paula, Roberto Pinheiro, Andy Postlewaite and Irina Telyukova for very helpful discussions. I am responsible for any errors.
1 Introduction

In the labor market, many workers and firms form relationships that are, in principle, meant to last a long time, and often they do. These relationships are not set in stone, however. In most cases either party can decide to leave the relationship at any time. For firms, the decision to leave a match (i.e. to dismiss the worker) could follow from an understanding that continued employment of the worker will no longer be profitable. Likewise, a worker could decide to leave a firm for alternative and better employment elsewhere, or to spend more time at home. Assuming that at least one of the parties likes the current employment relationship, i.e. it earns more in the match than it could by itself, it naturally worries about the participation of the other party.

In this paper, I study and characterize the labor contract offered by firms to workers, in a model with labor search frictions that takes seriously mutual uncertainty about the match. On the worker side, the uncertainty is his own productivity in the match - the worker may know, say, how hard he works, but does not know what value he yields to the firm and whether it is profitable to the firm to keep him. On the firm side, the uncertainty is the nature of outside offers that the worker may receive - the firm does not engage in outside offer matching (discussed below). At each point in time, each party’s participation decision has to take into account the expected participation decisions on the other side. What kind of a contract would entice a worker to take a job whose tenure he cannot be certain of? How does the firm go about retaining good workers? How long should it keep the bad workers, if at all? What wages should it pay over time? These are the questions that I aim to answer in this paper, in a framework that incorporates the bilateral nature of employment uncertainty explicitly, in the setting of a frictional labor market. (See the discussion of related literature in

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1In 1996, 35.4% of workers between age 35-64 reported their job tenure to be greater than 10 years. 20.9% of workers between 45-64 years reported a tenure greater than 20 years. (Farber (1999), based on CPS data)

2This is a standard employment relationship in the U.S. (unless there is any evidence to the contrary, this is the default relationship), so-called “employment at will”, where either party can terminate the relationship without notice for any reason, or for no reason at all. (Malcomson 1999, Malcomson 1997)

3Participation uncertainty matters in frictional labor markets. The frictions make it costly for the firm to find replacement workers. Therefore, worker turnover is often a major concern for firms. Moreover, employer to employer transitions occur frequently. Fallick and Fleischman (2004) document that 2.6% of workers move to another employer each month, and nearly 40% of the jobs started between 1994-2003 represent employer changes (based on CPS data). Likewise, from the worker side, the concern the worker often faces is the possibility of being let go, whether for external reasons (changes in the
the next section). It is important that we understand how an employment relationship may be characterized in the presence of this bilateral uncertainty, as it is likely to affect not only the relationship itself, but, more broadly, the entire labor market and the dynamics of aggregate phenomena such as worker turnover and displacement.

To be concrete, the model incorporates the following key features. First, search frictions imply that a worker and firm can be better off together than each of them by themselves. Finding an alternative match is costly, and hence there is generally a surplus associated with being matched. Search frictions create an environment in which the desire for long-term relationships is natural on both sides of any match. Second, there is bilateral participation uncertainty during the relationship. On the one hand, workers search on the job and meet alternative employment opportunities randomly. On the other hand, the value of production is observable only to the firm, and not to the worker, which makes the worker uncertain about the firm’s participation in the future. As the worker is not sure how valuable he is for the firm, he is uncertain how willing the firm is, at a point in time, to keep participating in the match.

Third, the firm commits to its wage contract over the employment relationship, conditional on the survival of the match. This is a stylized view, but one often taken in literature on frictional labor markets (see below). The contract specifies the amount that the firm pays the worker at each moment during the relationship, if both parties are still participating in the match. This wage schedule is posted and seen by every worker who meets the firm. Fourth, neither the worker nor the firm are obligated to stay in the match at any point in time. Finally, neither the worker nor the firm can respond when the other side decides to pull out of the match. This inability to respond when the participation constraint becomes binding on the opposite side captures an empirically relevant feature of labor relationships, and places the model in industry, say), or for reasons internal to his job that he may not be aware of (say, he cannot judge how valuable his work is for the firm). There are significant losses associated with displacement: dismissed employees often have to spend time in unemployment while looking for a new job, or have to be content taking a job that earns significantly less (Kletzer 1998).

4For the worker, if the firm decides to fire him, there is nothing he can do. For the firm, I assume that it does not engage in offer-matching. This assumption captures either the inability -perhaps caused by the difficulty of firms to see the outside offers of their workers- or the lack of desire of many employers in the data to engage in offer matching. Scott et al. (2004) find that nearly 80% of employers either did not make a counter-offer to any of their workers, or made a counter-offer to less than 5% of their workers, over the last year. (Their data is from the WorkatWork survey.) Weiss (1990), Mortensen (2003) and Manning (2005) discuss the reasons why firms prefer not to match.
a second-best world. Here, it is possible that parties walk away from relationships that are still creating surplus, and the constrained-optimal contract attempts to minimize such mistakes, while maximizing the overall payoff that falls to the firm -two conflicting motivations that have to be balanced out.

The firm aims to extract as much surplus as it can from each worker of any productivity. To keep the workers in the match, it wants -in general- to shift the date the compensation is paid out as much into the future as possible (this is known as backloading in the literature). At the same time, it values the participation of the high productivity worker more. Since the productivity of the match is unobservable to the worker, her participation depends, among other things, on the firm’s expected participation for each productivity, and the firm takes this into account. So postponing compensation till the end may backfire as workers (in the bad, but also in the good matches) would be too uncertain about the participation of the firm, and they will be tempted to leave for other offers. The result is that the firm uses the wage schedule strategically, to reveal information to the worker about her productivity in order to retain her, but it does not want to do so too soon.

The major result of this paper is that the optimal contract exhibits a ‘gradual’ wage growth. Moreover, the amount of wage growth over the course of the relationship depends, in general, on the magnitude of the bilateral participation uncertainty faced by both parties. The model’s features and outcomes can both speak to important observations in the data and they could tell us something new about these observations. For example, we do observe that wages generally increase over the tenure of a single job. At the same time we see turnover and dismissals decline over the employment relations, as in the model. However, at the same time, we can gain new insights about the dependence of the contract on the characteristics and magnitude of the uncertainty that the parties face. This would allow us to link the contract to forces in the U.S. economy that affect the participation decisions on both sides, to study aggregate phenomena in the U.S. job market. To do so, I am working on extending the current model to the market-wide case. I will discuss this extension in more detail at the end of the paper.

All in all, the major contribution of this paper is to propose an alternative theory of wage growth over the job tenure. To be sure, this is not the first paper to think about wage growth, but it is a new approach that takes seriously crucial features of the labor market that have not been studied jointly before. I contrast the paper with
alternative theories of wage growth in the next section.

The rest of the paper develops as follows. In the next section, I discuss the model in
light of the related literature. Section 3 sets up the model, focusing on the decisions each
party makes and defines the equilibrium. I derive the optimal contract in the section
4. In section 5, I discuss the result and its implications for more general settings. In
particular, I discuss the potential impact of more general forms of uncertainty on the
firm’s side on wage contracts and wage growth. I briefly describe the implications of
the model for aggregate phenomena and several extensions of this research agenda.
Section 6 concludes.

2 Related Literature

The model fits in the strand of economics that models search frictions in the labor
market explicitly. During the match, the worker can receive offers from other firms
at random times, as in the on-the-job search literature (Burdett 1978 and thereafter).
One of the standard ways in which the wage is formed in these models is posting: the
firm is assumed to be able to commit to whatever wage or contract it posts. In equilib-
rium models with on-the-job search, there are two basic standard procedures for wage
formation: wage bargaining (see, e.g., Pissarides 2000) and wage posting. I follow the
latter route, because it allows me to formulate the problem of the optimal intertemp-
oral division of payments over the relationship in a clean way. Moreover, I avoid the
complications of modeling bargaining in the presence of asymmetric information. Bur-
dett and Mortensen (1998) solve for the market equilibrium in the setting with wage
posting and on-the-job search. However, they restrict the wages to be constant over
the entire relationship. Shimer (1996) noted that in general, a constant wage is not the
optimal policy in models with on-the-job search.

Stevens (2004) solves for the optimal wage-tenure contract (with time-varying wages)
in a market where risk-neutral workers search on the job. Her result is that it is optimal
for firms to engage in extreme backloading: the firm pays the minimum it is allowed to
pay the worker for some time, and switches to paying the worker his productivity there-
after. Burdett and Coles (2003) solve a similar case, but, crucially, with risk-averse
workers. Risk aversion drives their labor contract: wages increase gradually over the
course of the employment relationship, but purely as a result of the desire to smooth
consumption over time.
In neither Stevens, nor Burdett and Coles, is there any uncertainty about the firm’s ability or willingness to pay up the large amounts it is promising later in the relationship. I, on the other hand, incorporate the notion that these promises are only as good as far as they are in the firm’s interest to keep: a firm could promise to pay a lot at a much later date, and then simply fire the worker before the date comes. Moreover, the worker is uncertain about what exactly the firm’s interest in the relationship is. In such an environment, it is much harder to sustain an employment relationship. One of the contributions of this paper is showing that uncertainty about the firm’s participation can mean that extreme backloading is no longer optimal, even with risk-neutral workers and firms. To the best of my knowledge, (ex post) uncertainty about the quality of the match or firm, or about the firm’s ability to pay wages in the future, has not been incorporated in models with on-the-job search (with the exception of Postel-Vinay and Turon (2006), discussed below).

Two important elements of the model are that the firm cannot match outside offers and the firm cannot write a contract contingent on the match-specific productivity shock. This restricts the ability of the parties in the match to adjust to binding participation constraints on one of the sides. Following Burdett (1978), Mortensen (1989) and Burdett and Mortensen (1998), there is an entire strand of literature that takes the approach of not matching counteroffers (see Mortensen 2003 for an overview). If the firm and the worker could respond each time the participation constraint on the other side became binding, the future participation consideration would largely be absent in the determination of the current contract, as both parties would know that the contract can adjust to accommodate participation. An important dimension of the participation uncertainty would be eliminated. This is the case that Postel-Vinay and Turon (2006) study (with productivity shocks), following work of Postel-Vinay and Robin (2003, 2005) with heterogeneous firm productivities.

In contrast to Postel-Vinay and Turon, the model in this paper is set in a second-best world, where a worker who leaves the match does not take into account that it destroys the profit of the firm, and the firm does not take into account the value of employment that it destroys when firing the worker. The optimal contract in this paper trades off, ex ante, the participation risks after the match is formed. The major outcome is that wages rise over the course of the relationship, as an optimal response to these risks. Moreover, wages can rise relatively early, because the raise contains information (inferred in the equilibrium): if the match was bad, the firm would have
fired the worker. It is the tension between backloading to prevent turnover and giving early raises to reveal information that provides the rationale for gradual wage growth on a job.

Of course, there are theories that can account for wage growth. One such theory is based on Jovanovic (1979). Here, workers and firms are uncertain about the quality of the match, and will gradually find this out over the duration of the relationship. Good matches will survive longest. The major difference between this theory and the present model is that it is not exogenous release of information that drives wage growth in the model in this paper, but rather the endogenous incentives of one informed party to tell the uninformed party about the quality of the match.

Perhaps the best-known theory of wage growth is the theory of human capital (Becker 1962). Not all observations on the labor market can be easily accounted for in the pure human capital story (in its basic forms). General human capital (transferable across employers) for example, is thought to explain wage growth with labor market experience. However, displaced workers often face prolonged losses of income. Firm-specific human capital can be an alternative explanation, but the mechanism by which skills that are not valued at other firms, and not priced by the market, are rewarded, is unclear. It is often imposed that there is a certain risk that outside offers of varying quality arrive during the relationship (see Manning’s 2005 discussion) to force the firm to increase wages for a worker with firm-specific human capital at least partially. However, in a setting with on-the-job search, as illustrated in this paper, wage growth can occur without firm-specific human capital. Of course, it is an interesting question to distinguish empirically between these causes of wage growth, one that has spawned a large literature, and no general agreement. Thus, while this paper does not want to downplay the importance of human capital, it also holds that there is room for proposing and investigating alternative theories.

Moreover, a number of papers (Medoff and Abraham 1980, 1981, Kotlikoff and Gokhale 1992 and Abowd and Dygalo 2005) have found that productivity and wages do not rise in a proportional relation, instead the older workers are paid a higher fraction of their productivity, which appears to support that there are other motivations than productivity alone for the observed wage profiles.

One only needs to look at the discussion on the returns to tenure, for example.
3 Model

In the current model I do not take all causes of job insecurity head on; instead I opt for a stylized view, focussing on one aspect of the uncertainty that the worker faces about the firm. The worker cannot see the value of his production in the match, and this value is an important determinant in the participation decision for the firm. This allows us to capture a key trade-off between facilitating the worker’s and the firm’s participation, while at the same time it sets up the foundation for applying the model to other sources of job insecurity.

3.1 Setup

A measure 1 of firms and an identical measure of workers live in continuous time. Workers and firms meet at rate $\lambda$. Before a meeting occurs, a firm posts a wage-tenure contract $w(t)$; a firm is committed to pay this wage as long as the worker-firm match lasts. However, either party may leave the match at any time. Workers keep receiving outside offers while being matched, at rate $\lambda$, and drawn from a distribution $F(V)$ which is bounded from above.

Directly after the match is formed, a productivity of the match is drawn from a distribution $G(p)$ with bounded support, and observed by the firm, not the worker. Throughout the paper I focus predominantly on the case with a countable number of productivities. Given the posted wage-contract, the productivity determines how large the amount is that the firm is residual claimant of. However, given a non-stationary wage policy, the firm might find itself in the position that a previously profitable relationship becomes unprofitable. In this case, the firm will walk away from the relationship, but only at the moment that it is better to do so\(^7\)

Limited commitment to on-going participation has a bite for the following assumption: firms cannot observe the offers that workers get while they are employed at the firm. As noted before, I assume that the firm will not engage in counteroffers. Weiss

\(^7\)I restrict myself here to ‘simple’ wage-tenure contracts, i.e. for each tenure $t$ there is a single wage specified. Moreover, I assume that commitment is complete, in the sense that the firm always has to pay this amount whenever the worker works for him. Alternatively, one can focus on renegotiation proof contracts as well. This issue will be addressed as part of the further research agenda. Suffice to say that in many firms the pay-structure is to standardized enough to make the ‘simple wage-tenure contracts’ relevant. If one wants to have a concrete example of contracts that look like the contracts covered in the paper, one can think of up-or-out contracts, for example.
(1990) and Mortensen (2003) give a list of rationales why firms are reluctant to engage in offer-matching: workers can decide on the intensity of search for outside offers, and the firm wants to discourage searching, the firm does not have any practical verification mechanism for unobservable outside offers, and there are problems associated with unequal wages for identical jobs (the high wage guy gets fired first; or behavior of other workers in joint production is affected). All in all, the restriction on offer matching can be seen as capturing a relevant dimension of labor relations in a significant part of the labor markets. As a result, workers might very well leave the firm when the firm is still making positive profits. In other words, the match break-up is not necessarily privately efficient between the worker and the firm.

Moreover, on the firm’s side, when wages become too high, the firm will have to fire the worker in a low productivity match, even though this worker is better off in the relationship at the current wage than in unemployment. All in all, the firm’s and worker’s possibility to leave the other party behind while that party is still earning a surplus, create a central tension in the wage-schedule posted: too high wages will make the worker face considerable risk of being laid off, whereas low wages may lead the worker to quit his job quickly.

Workers and firms are both risk-neutral. However, I assume that the firm cannot sell the job in advance: at any time the wage has to be above some level $c$, which could be the minimum wage of a country, or the subsistence level of a credit-constraint worker. Finally, I impose an exogenous death rate $\delta$ on the worker. Matches in the model are therefore terminated either by an exogenous death, after which the worker is replaced by an unemployed worker, or by the firm or worker. If the firm terminates the match, the worker becomes unemployed again.

To study a labor market with long-term relationships and bilateral participation uncertainty, there are two steps to be taken. First, and this is the focus of the paper at hand, I study the behavior within the match. In particular, I derive the optimal contract for the firm, taking the offers of other firms as given. Following this, the

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8I also rule out severance payments after firm-induced match-breakups, to make the worker-firm more resemble ’employment-at-will’. This would nevertheless be an interesting consideration, since it could distinguish the theory in the paper from other theories. A couple of authors have found that increases in job security (because of severance restrictions or payments) have led to more steeper wage-profiles. See e.g. Neumark and Stock (2001) and Osuna (2005), which is exactly what the model would predict. Note that models of on-the-job search with risk aversion make a similar assumption as the no-sale condition on the slope of the utility function, to rule out selling of the job.
natural next step is to focus on the resulting market equilibrium, where each firm posts a contract of certain values to the worker, and acceptance probability of this contracts depends on the steady state distribution of workers over employment status and tenure.

3.2 Worker-Firm relationship

Let us concentrate on the relationship between a worker and a firm, who have just met, and where the firm previously has posted a wage-tenure contract with a life-time expected value $V_0$. The value $V_0$ is an equilibrium object that summarizes the life-time utility that a worker can expect at $t = 0$ when he accepts the contract. It incorporates the payments of the firm to the worker, but also the outside offers that the worker finds profitable to take, and the unemployment benefit the worker gets in case he becomes unemployed.

The timing is as summarized in figure 1. The firm’s problem is to maximize expected profit, subject to the constraint of a ‘simple wage-tenure contract’ as defined above with should provide a value $V_0$ to the worker at the beginning of the match. The firm knows that the wages it promises are discounted by the probability that the match will not survive up to the point. The total survival probability of the match depends as much on the expected probability that the worker has left the firm for a better opportunity elsewhere, as the expected probability that the firm will have found it better to end the match beforehand.

To be clear, when speaking of “ex ante” below, I am referring to the time of the contract posting, before the productivity has realized.

Note, that the restriction that the firm cannot see the productivity before the match is realized is not without loss of generality. It rules out that the firm uses the wage contract to signal the quality to the worker.
3.3 Decisions and Strategies

The firm takes two kinds of decisions. First, it decides to post a wage-tenure contract \( w(t) \). Secondly, it has to decide at any point during the relationship whether to continue the match, or to walk away. The latter decision depends, of course, on the productivity \( p \) that has materialized, and the wage-tenure contract that the firm has posted before. It also depends on the firm’s expectations about the worker’s acceptance behavior. The first decision, which contract to post, depends on the firm’s ex ante expectation over the probability distribution of productivities that can realize at the beginning of the match. However, when posting the contract, the firm’s expectation about the worker’s behavior in accepting, and staying or leaving during the match given the posted contract, plays an equally important role.

The worker takes decisions about participation during and before the match. First, during a match, he constantly weighs whether he wants to continue or walk away. At a moment that he receives an outside offer, he compares the value he attaches to the outside offer, with the value he attaches to the continuation of the current relationship. To calculate the value of continuation, the worker needs to have expectations of the (probability of a) productivity in the match, and the future participation decisions of the firm. Furthermore, when the value of the match drops below the value of being unemployed, the worker will also leave the match. Lastly, before starting his employment, the worker bumped into the firm, made the same calculation whether to accept or reject the offer, but based on the value of in previous (un)employment.

Let \( a_f(t) \) be the decision to continue in the match at time \( t \) for the firm, conditional on match survival up to that time. Then a strategy for the firm is the following:

- a wage-schedule \( w(\tau) \in \text{the space of functions with domain } [0, \infty] \)
- a participation decision \( a_f(t, p, w(.)) \rightarrow [0, 1] \),

where one could in principle allow from continuation with some intermediate probability. Note that with \( w(.) \), I mean that the entire wage-tenure schedule is an argument in the strategy of the firm.

Let \( a_w \) denote a decision for the worker to continue with match. Then a strategy for the worker is a set of continuation decisions:

- an acceptance decision during the relationship when no outside offer is considered, i.e. whether to quit to unemployment or not: \( a_w^u(t, w(.)) \rightarrow [0, 1] \);
• an acceptance decision during the relationship when there is an outside offer of value \( \hat{V} \) is considered: \( a_{w}^{u}(\hat{V}, w(.), t) \rightarrow [0, 1] \);

When looking at the equilibrium in the entire market, there is a third acceptance function:

• an acceptance decision before the relationship, upon observing a wage schedule \( w(.) \), \( a_{w}^{x}(w(.)) \rightarrow [0, 1] \);

However, when focussing on the behavior within the match, I replace this dimension by the requirement that \( w(.) \) provides an expected value of \( V^0 \) or more to the worker.

As said, the value for the worker of staying in the relationship evaluated at time \( t \), the current value \( \tilde{V}(t) \), is an equilibrium object, given wage schedule \( w(.) \), and expectations about continuation actions on both sides. To keep notation simple, I will not index \( \tilde{V}(t) \) by the equilibrium strategies. The tilde indicates that \( \tilde{V} \) is a current value, evaluated at time \( t \). For simplicity let us assume that a worker will always break up the current match as long as the value of the outside offer is weakly higher than the value he attaches to staying in the relationship\(^{10} \). Optimal continuation decisions are therefore

\[
a_{w}^{u}(\hat{V}, w(.), t) = 0, \text{ iff } \tilde{V}(t) < \hat{V}
\]

\[
a_{w}^{u}(w(.), t) = 0 \text{ iff } \tilde{V}(t) < V^u
\]

With an arrival rate of \( \lambda \) of outside offers, the rate at which the worker leaves the match is then \( \lambda(1 - F(\tilde{V}(t))) \) at each \( t \). This implies the following survival function \( \psi(t) \) for the continuation decisions of the workers., i.e. the probability that the match is still intact at time \( t \), given \( \tilde{V}(t) \) is given implicitly by

\[
\dot{\psi}(t) = -\left[\lambda(1 - F(\tilde{V}(t)))\right]\psi(t),
\]

which reduces to the following \( \psi(t) \):

\[
\psi(t) = e^{-\int_{0}^{t}\left[\lambda(1 - F(\tilde{V}(\tau)))\right]d\tau}, \quad \psi(0) = 1,
\]

as long as \( \tilde{V}(\tau) > V^u \), \( \forall \tau \in [0, t] \), otherwise \( \psi(t) = 0 \).

Let us assume for now that each productivity will be broken at a unique time (or never at all),

\[
\hat{t}(p_i) = \arg \min_{t} a_{f}(t, p_i, w(.)) = 0.
\]

\(^{10}\text{From the analysis below it becomes clear that this assumption is not material.}\)
In any case, it is straightforward to show that whenever it is optimal to keep productivity $p_i$, it is optimal to continue matches with $p > p_i$. Hence, at any time the firm keeps all productivities higher than the lowest productivity in the match. Define therefore $\bar{p}(t)$ to be this lowest productivity still in the match. The survival function of the firm in the match is therefore at each $t$

$$[1 - G(\bar{p}(t))].$$

Now, everything is in place to look at value of being in the match, for both the worker and the firm.

### 3.4 Value Functions for Firm and Worker

One can define the value of a wage policy for the worker at time 0, and the current value $\tilde{V}(t)$, at each time $t$, given that the worker has (in equilibrium correct) expectations about $\psi, \tilde{t}, \bar{p}$, and takes $F(\tilde{V}), V^u, G(p)$ as given. Let $g(p_i)$ be the probability that productivity $p_i$ is drawn at the beginning of the match.

$$V(0) = \int_{0}^{\infty} e^{-\delta t}\psi(t)[1 - G(\bar{p}(t))] \left( w + \lambda \int_{\tilde{V}(t)} \tilde{V} dF(\tilde{V}) \right) dt + \sum_{p_i=p}^{\bar{p}} e^{-\delta t(p_i)} \psi(\tilde{t}(p_i))g(p_i)V^u$$

To understand this value function, consider the term $\{1\}$: this is the probability that a match is still together at time $t$. Term $\{2\}$ is the flow that the worker gets at each instant: he gets $w(t)$, and with probability $\lambda$ he gets an outside offer. Upon accepting an outside offer, he gets the value of the outside offer, which I here, wlog, treat as if he consumes the outside offer instantly, in its entirety. In the next instant, $\psi(t)$ is decreased, because the match did not survive. The last term, $\{3\}$, incorporates probability of being dismissed and the resulting continuation value of unemployment at that time.

Along the same lines as $V_0$ one can derive current value $\tilde{V}(t)$. One has to correct for the fact that time index $t$ is different, and that the remaining productivity distribution is different, if there were dismissals before $t$. Having said this, $\tilde{V}(t)$ is constructed
exactly in the same way as $V(0)^{11}$:

$$
\tilde{V}(t) = \int_{t}^{\infty} \left( e^{-\delta(\tau-t)} e^{-\int_{t}^{\tau} \lambda(1-F(\tilde{V}(\varsigma)))d\varsigma} \frac{1-G(\tilde{p}(\tau))}{1-G(\tilde{p}(t))} \left( w(\tau) + \lambda \int_{\tilde{V}(\tau)}^{\hat{V}} dF(\tilde{V}) \right) \right) d\tau 
+ \sum_{\mu_i > \tilde{p}(t)} e^{-\delta(\tilde{p}(\mu_i)-t)} e^{-\int_{t}^{\tilde{p}(\mu_i)} \lambda(1-F(\tilde{V}(\varsigma)))d\varsigma} \frac{g(\mu_i)}{1-G(\tilde{p}(t))} V^u, \tag{6}
$$

Note the following,

$$
e^{-\int_{t}^{\tau} \lambda(1-F(\tilde{V}(\varsigma)))d\varsigma} = \frac{\psi(\tau)}{\psi(t)},
$$

and naturally, $e^{-\delta(\tau-t)} = e^{-\delta t}$, so the following relation holds

$$
V(t) = \psi(t)e^{-\delta t}[1-G(\tilde{p}(t))]\tilde{V}(t),
$$

where $V(t)$ is the part of $V^0$ that realizes after time $t$. One can see that the value which a worker assigns to a wage schedule $w(\cdot)$ depends on the wages the firm promises to pay, the probability that these wages are actually paid (because the match might break up exogenous, by the firm or by the worker himself), the outside offers that the worker can take, and the probability and value of becoming unemployed. A higher wage schedule raises $\tilde{V}$ and naturally increases the participation. However, this wage schedule is in a sense discounted by the firm’s participation decisions. This can be seen clearly in the above equation, where the two factors in the value function that are directly determined by the firm, are

$$
\frac{1-G(\tilde{p}(\tau))}{1-G(\tilde{p}(t))} \cdot w(\tau). \tag{7}
$$

Thus, as the worker is more certain that he will get the wage payment (i.e. the first part of the term above is closer to one), a given wage-schedule will raise the current value more. Thus, the firm has two ways of creating a high $\tilde{V}$: higher wages, or a higher probability that they are paid.

Note that in the case of discrete productivities, the value for the worker who remains in the match will jump each time he expected the firm to fire a certain productivity. It tells him that he is of higher productivity, and that future rewards are now more likely.

Let us now focus on the profit for the firm. Given a productivity $p$, worker-survival function $\psi(t)$, wage schedule $w(\cdot)$, and $\tilde{t}(p)$ the conditional profit function is given by

$$
\tilde{\Pi}(p, t) = \int_{t}^{\tilde{t}(p)} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} (p - w(\tau)) d\tau \tag{8}
$$

\(^{11}\)Where I take the following stand on dismissals: if at time $t$ a productivity is fired, I take $\tilde{V}(t)$ to be the current value after dismissals would have taken place, thus incorporated in $\tilde{V}$ is that the worker \textit{knows} it is not one of people being fired at time $t$. 

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The optimal continuation profit he can get from that moment onwards.

\[
\Pi(p, t) = \max_{t'} \int_{\hat{t}(p)}^{t'} e^{-\delta(t-t')} \frac{\psi'(\tau)}{\psi(t)} \left( p - w(\tau) \right) d\tau \leq 0. \tag{9}
\]

The dismissal time \( \hat{t}(p) \) has to satisfy the following: \( \Pi(p, \hat{t}) \leq 0 \), and for no \( \hat{t} < \hat{t}(p) \) it is the case that \( \Pi(p, \hat{t}) < 0 \). Thus, if at any time \( t \), the current value expected profit, i.e., the expected discounted profit at that time (conditional on surviving up to that point, and conditional on taking weakly optimal break-up decisions afterwards), is at zero, or strictly below zero, then the match can, respectively will be, terminated by the firm. From the equations above it follows immediately that \( \hat{t}(p) \) is indeed an increasing function of \( p \). Likewise \( \hat{p}(t) \) is increasing.

Now, let us look at the ex ante stage, for the firm, where the firm evaluates which contract to post, ex ante profit for a wage schedule \( w(\cdot) \), where I take \( \psi(t), \hat{t}(p) \) or \( \hat{p}(t) \) from the appropriate subgame that follows the posting of a contract \( w(\cdot) \). \( \Pi^{\text{ex ante}} \) is then given by\(^{12}\):

\[
\Pi^{\text{ex ante}} = \int_0^{\infty} e^{-\delta t} \psi(t) \left( \sum_{p_i = \hat{p}(t)} (p - w(t)) g(p) \right) dt \tag{10}
\]

Define, for future reference, the \textit{ex ante expected current value (ecv)} of profit as well:

\[
\Pi^{\text{ecv}}(t) = \int_t^{\infty} e^{-\delta \tau - t} \frac{\psi'(\tau)}{\psi(t)} \left( \sum_{p_i = \hat{p}(\tau)} (p - w(\tau)) g(p) \right) d\tau \tag{11}
\]

### 3.5 Equilibrium Definition

Let us define the Within-Match Perfect Equilibrium

**Definition 1** The Within-Match Perfect Equilibrium is a set of functions \( a_i(\cdot), \psi(\cdot), \hat{p}(\cdot), \hat{t}(\cdot), \hat{V}(\cdot), V(\cdot), \Pi^{\text{ex ante}}, \Pi, w(t), F(\hat{V}), G(p), \) and \( V^u \) such that

1. **Second Stage:** Workers take optimal continuation decisions given \( \hat{V}, V^u \) at any point during the relationship; \( \psi \) follows from these decisions of workers, through \( (3) \) and \( (4) \).

\(^{12}\)Or, alternatively by

\[
\Pi^{\text{ex ante}} = \sum_{p_i = \hat{p}} \left( \int_{0}^{\hat{t}(p_i)} e^{-\delta t} \psi(t)(p - w(t)) dt \right) g(p)
\]
2. Firms take optimal continuation decisions, given $\Pi(t,p)$ as defined by (9), and the productivity $p$ at any point $t$ during the relationship. The breakup functions $\tilde{p}, \tilde{t}$, and survival function $(1 - G\tilde{p}(t))$ follow from the optimal continuation decisions of the firm.

3. (“Rational Expectations”) The value functions $\tilde{V}, V, \Pi^{ex \ ante}, P_i, \tilde{\Pi}$ follow from $\psi, \tilde{p}, \tilde{t}, (1 - G\tilde{p}(t)), w(\cdot)$, given $F(\tilde{V}), G(p), V^u$ through equations (5),(6),(9),(10).

4. First Stage of the game: $w(\cdot)$ is chosen, given $\psi(t), \tilde{t}(p), V, \tilde{V}, \Pi, \tilde{\Pi}$ that satisfy 1.-3. where the posted $w(\cdot)$ will be taken as given. Ex ante profit is constructed as in (10), and is maximized over the space of functions and subject to $V(0) \geq V^0$.

where $F(\tilde{V}), G(p)$ and $V^u$ are outside the match and taken exogenously.

Note that the restriction to subgame perfect equilibrium implies the usual; concretely, I rule out equilibria where agents take non-optimal actions after some deviating wage contract $w_d(\cdot)$ is posted. I require that any $w(\cdot)$ is evaluated with functions $\psi(t), \tilde{t}(p)$, that are optimal in the subgame, i.e. satisfy conditions 1. through 3. This makes sure, e.g., that I don’t have an equilibrium in which all workers in the economy will non-credibly threaten to leave the firm immediately, unless they get a contract that guarantee a certain value $V$.

3.6 Surplus of a Match

It is often easier to work with the joint surplus of the relationship, instead of looking separately at the value of the relationship to the firm $\Pi(t)$, and to the worker $V(t)$, or perhaps better, $V(t) - V^u$, the net value of being employed at the firm over being unemployed. Note that the assumptions on market and matching behavior imply that the firm can hire anybody who bumps into the firm. Thus, the outside option for the firm is zero. The value of unemployment is stationary, and can be derived as

$$\delta V^u = b + \lambda \int_{V^u} (\hat{V} - V^u) dF(\hat{V}).$$

Let $J^{ecv}$ the expected current value of the surplus. It can then be shown that the value of the surplus is

$$J^{ecv}(t) = \Pi^{ecv} + [1 - G(\tilde{p})(t)](V(t) - V^u)$$

$$= \int_t^\infty e^{-\delta \tau} \sum_{p_i = \tilde{p}(\tau)} \psi(\tau) \left( p_i - b - \lambda \int_{V^u} (\hat{V}(\tau) - V^u) dF(\hat{V}) \right) g(p_i) \ d\tau, \quad (12)$$
where the derivation of (12) can be found in the appendix. Note that, if one could hold
the participation decisions of workers and firms constant, one could achieve any division
of the current surplus: \( w(t) \) does not turn up directly. However, \( w(t) \) influences the
surplus through the participation decisions. Thus at any moment \( t \), future \( w(\tau), \tau > t \)
influences the participation of both firm and worker. The problem of the firm is to
design the optimal contract that allows it to capture as much of the surplus as possible,
taking the participation decisions into account.

4 The Optimal Contract

I approach the problem of finding the optimal contract in the following way. First, I
show that an optimal response has certain properties, and then I limit the problem
of finding optimal responses to the set of policies that has this property. After these
reductions of the set of responses, I will solve for the equilibrium, within this set. The
equilibrium outcome of this restricted problem is an equilibrium outcome in the original
game as well.

The first result is an optimal contract always takes a particular form:

**Proposition 1** An optimal contract will have wages that are weakly increasing over
time; the optimal quit decision therefore occurs at \( w = p \). (Or, if wages jump up, at the
first time that \( w > p \).)

**Proof** See the appendix. ■

Thus, even though there is a participation risk on both sides, the optimal contract has
increasing wages. This is due to the different nature of the participation constraints
on the two sides. The factor behind the firm’s participation constraint is persistent,
whereas the worker’s outside options arrive at random moments and are randomly
drawn each time. Thus, at each moment the worker might leave, and the fact that he
has stayed up to now, does not say anything about tomorrow’s chances of him leaving.

The random arrivals of outside offers, without persistence, makes the firm want to
postpone wages\(^{13}\). The intuition can perhaps be illustrated as follows: suppose a firm
has productivity 20. A worker works two periods for the firm. After the first period
an outside offer arrives, which is 8 with probability 0.5, and 18 with probability 0.5.
Suppose the firm originally gives an equal wage 10 in each period. The firm can do

\(^{13}\)This motive appear e.g. in Harris and Holmstrom (1982), Burdett and Coles (2003), and Stevens
(2004).
better by postponing. Note that originally with probability 0.5 the worker will leave
the firm. Now keep the probabilities constant, and backload as much as necessary,
keeping constant the total wages provided over the times that the worker originally
wanted to stay. This means paying wage 18 in the last period, and 6 in the first period.
Revealed preferences (1): The worker will (weakly) prefer this: he will get 6+18=24
out of this relationship, the same as he got before when he would leave in 50% of the
cases. The firm does strictly better. Revealed preferences (2): the firm pays the same
amount over the realizations where the worker would stay originally, but now also has
to pay when the worker chooses to stay where he did not before. But this is good for
the firm: he now pays 18, keeps 2, where he would not have anything before. So, both
worker and firm are better off, and the firm is strictly better off.

Offsetting the benefit of delayed wage increases is the fact that the firm might
dismiss the worker, when the time comes for the wage raise. Dismissing the low-
productivity worker could still be a policy that maximizes the (participation con-
strained) surplus, ex ante. In this case, the firm can promise more to the worker
(in expectation), and therefore postpone more. On top of this, it could be possible
(shown below in detail) that the firm wants to raise wages before, the inform the high
productivity worker that he is in a good match, thus allowing the firm to postpone
wages more for the high-productivity worker, after the firm’s participation uncertainty
is resolved or diminished. The latter is exactly the “conditional effect” in (7). The
firm ‘informs’ the worker by raising wages to a level at which it is only profitable to
keep the worker when the match is good. The persistence of the match productivity
is important for the firm: it gives the firm both the motivation and possibility to raise
the wages and make the argument to the worker in the high-productivity match. The
remaining issue is what the best way is for the firm to do this. The crucial observation
is that the worker knows that profitability of the low-productivity worker has fallen
to or below zero. But this puts no direct requirements on shape of the future wage-
schedule (i.e. whether it has to be upward or downward sloping, monotone, etc.); in
fact, the firm still wants to backload as much as possible. What is the best way to tell
the worker with minimum amount of frontloading: a wage that is raised, but then is
as flat as possible, constant perhaps. Thus, the wage schedule will still be increasing.

Let us now proceed to characterize the contract more precisely. Since \( \bar{p} < \infty \), I
can restrict or focus on bounded wage-schedules as well. Since the optimal wage policy
is monotone, it follows that \( w(.) \) is a.e. differentiable. Following the lead of Stevens
one can construct right-differentiable (recursive) value functions between dismissal times, taking the values at the points of dismissal as boundary points.

Consider the case of two productivities, \( p_l, p_h \), where \( p_h \) arises with probability \( \alpha \). Time derivatives are denoted with a super-scribed dot, and the current value is denoted by a superscript \( cv \)-term. The expected current value, from an ex ante perspective (i.e. before the match productivity has realized) is indexed by \( ecv \).

**Firm’s Profit** Let us focus on the firm’s profit first. The (ex ante) expected current value at time \( t \), where \( t \in [0, \tilde{t}(p_l)] \) is given by

\[
\dot{\Pi}_{ecv}(t) = (\delta + \lambda (1 - F(V_{cv}(t))))\Pi_{ecv}(t) - \alpha p_h - (1 - \alpha)p_l + w(t) \tag{13}
\]

This function tells us, recursively, the expected current value of profit at each time. At dismissal time \( \tilde{t} \) (since there will be only one dismissal moment in the model, I simplify the notation), I define two values for profit: one in expected terms, before the uncertainty about the match quality is resolved, \( \Pi^- \) and one after the uncertainty is resolved \( \Pi^+ \) (and the match is \( p_h \)), as follows

\[
\Pi^- (\tilde{t}) = \int_{\tilde{t}}^{\infty} e^{-\delta(\tau-\tilde{t})} \frac{\psi(\tau)}{\psi(\tilde{t})} (\alpha(p_h - w(\tau))) d\tau, \tag{14}
\]

and

\[
\Pi^+ (\tilde{t}) = \int_{\tilde{t}}^{\infty} e^{-\delta(\tau-\tilde{t})} \frac{\psi(\tau)}{\psi(\tilde{t})} (p - w(\tau)) d\tau; \tag{15}
\]

it is clear that \( \Pi^- = \alpha \Pi^+ \). After dismissal, the then current value is given by:

\[
\dot{\Pi}_{cv}^h(t) = (\delta + \lambda (1 - F(V_{cv}(t))))\Pi_{cv}^h(t) - p_h + w(t) \tag{16}
\]

This can also be put in ex ante expected terms, by taking the time derivative of the ex ante profit function (following Stevens) between the dismissal times

\[
\dot{\Pi}_{ecv} = \alpha \dot{\Pi}_{cv}(t) = (\delta + \lambda (1 - F(V_{cv}(t))))\alpha \Pi_{cv}(t) - \alpha(p_h - w(t)) \tag{17}
\]

---

\(^{14}\)Stevens refers in turn to Van den Berg (1990), in particular to his theorem 1. There are some technicalities relevant here; to be comprehensive this will appear in an appendix in the next version of the paper.

\(^{15}\)The reasoning that I go through below applies just as well to any countable number of productivities, but the two productivity case is notionally simple, because there are only two phases: the interval before the worker in the low productivity match is fired, which coincides with the ex ante perspective, and the period after that worker is fired, in which all uncertainty on the firm’s side is resolved.
Perhaps the signs in the recursive formulation can come across as counter-intuitive. This is the formulation basically tells us how a current promise of, e.g. $\Pi_{hv}^c(t)$ is delivered (i.e. the promise at time $t$ is held constant): if the risk of match break-up today is $(\delta + \lambda(1 - F(V_{hv}^c(t))))$, where the firm will lose today’s promise $\Pi_{hv}^c(t)$, and he gets a flow income of $p_h - w(t)$, it better be the case that the future promise makes up for any difference between the flow income, and the loss. Thus $\dot{\Pi}$ is increasing in the loss $(\delta + \lambda(1 - F(V_{hv}^c(t))))$, and decreasing in the flow income.

**Worker’s Value** I can repeat the same exercise for the worker, using the following boundary conditions:

$$V^{-}(\tilde{t}) = \alpha \int_{t}^{\infty} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} \left( w(\tau) + \lambda \int_{\tilde{V}(t)}^{\hat{V}(\tau)} \hat{V} dF(\hat{V}) \right) d\tau + (1 - \alpha)V^u,$$

and, likewise

$$V^{cv}(\tilde{t}) = V^{+}(\tilde{t}) = \int_{t}^{\infty} e^{-\delta(\tau-t)} \frac{\psi(\tau)}{\psi(t)} \left( w(\tau) + \lambda \int_{\tilde{V}(t)}^{\hat{V}(\tau)} \hat{V} dF(\hat{V}) \right) d\tau. \quad (19)$$

Current value for workers inside both intervals:

$$\dot{V}^{cv}(t) = \delta V^{cv} - w(t) - \lambda \int_{V^{cv}}^{\hat{V}^{cv}} (\hat{V} - V^{cv}) dF(\hat{V}), \quad (20)$$

where before $\tilde{t}$, $V^{cv}(t) = V^{ecv}(t)$. Alternatively, I can write the latter equation as

$$\dot{V}^{cv}(t) = (\delta + \lambda(1 - F(V^{cv}(t)))) V^{cv} - w(t) - \lambda \int_{V^{cv}}^{\hat{V}^{cv}} \hat{V} dF(\hat{V}), \quad (21)$$

**Surplus** Between $t = 0$ and the time that the worker in the low-productivity match is fired, the expected surplus follows

$$\dot{J}^{cv}(t) = (\delta + \lambda(1 - F(\hat{V}))) J^{cv} - (\alpha p_h + (1 - \alpha)p_l + b + \lambda \int_{V^{cv}}^{\hat{V}} (\hat{V} - V^{cv}) dF(\hat{V}) \quad (22)$$

I can rewrite this equation in simpler terms, without unemployment, and where I define $\Pi V \equiv \Pi^{ecv} + V^{ecv} = J^{ecv} - V^{u}$, for any $t$ before dismissal time; and $\Pi V \equiv \Pi^{cv} + V^{cv} = J^{cv} - V^{u}$, for any $t$ afterwards.

$$\dot{J}^{cv}(t) = \delta \Pi V - (\alpha p_h + (1 - \alpha)p_l - \lambda \int_{V^{cv}}^{\hat{V}} (\hat{V} - \Pi V) dF(\hat{V}) \quad (23)$$

From this formulation the private inefficiency is clear: the worker takes an outside offer whenever it is higher than is own value, but disregards the profit of the firm destroyed
in the process. After low productivity worker is fired, the current value of the surplus becomes, conditional on a continuing match:

\[
\dot{J}^{cv}(t) = \delta \Pi^V(t) - p_h - \lambda \int_{V^{cv}} (\dot{V} - \Pi^v) dF(\dot{V})
\]  

(24)

The boundary conditions, once again, are given by

\[
J^- = \Pi^- + V^- - V^u,
\]

and

\[
J^+ = \Pi^+ + V^+ - V^u
\]

Now, I have everything in place to study the shape of the contract offered in equilibrium.

**Lemma 1** Given feasible boundary points \(V^0\) at time 0, and \(V^-\) at some given equilibrium dismissal time \(\hat{\tau}\), a step-contract which has wages at either \(c\) or \(p_l\) is optimal for the firm. Likewise, for a contract after \(\hat{\tau}\) with initial value \(V^+\), a step contract is optimal, with wages at either \(p_l\) or \(p_h\).

**Proof** Note that little freedom is given here: the time interval and the boundary values are fixed. From proposition 1, we know that wages will never decrease. Given that \(\hat{\tau}\) was an equilibrium decision, it cannot be the case that it is better to lower below \(p_l\) during the second period, because that would yield a profitable deviation to delayed dismissal.

Let us concentrate on the first part of the lemma, the second part follows completely analogous. Index the step contract by \(s\), the generic contract by \(g\). The fact that the generic contract is feasible tells us that there is a wage schedule with \(c \leq w \leq p_l\), for any \(t\) in our interval. Then, a constant wage of \(c\) over the interval would lead to \(V(0) \leq V_0\), while a constant wage of \(p_l\) would lead to \(V(0) \geq V_0\). By continuity of the value in the stepping time, there exists a step contract that is feasible (i.e. respects the boundary conditions). I want to show that it is contract that gives most to the firm. First, note that at any time \(0 < t < \hat{\tau}\), \(V_s(t) \geq V_g(t)\) (strict if \(w_g(s) \neq c, w_g(s) \neq p_l\) for some interval \([s_1, s_2]\): suppose not: \(V_s(\hat{\tau}) < V_g(\hat{\tau})\), but then, if \(w(\hat{\tau}) = c\), \(V_s(0) < V_0\), because \(\dot{V}_s \leq \dot{V}_g\) (strict if there is a strict difference in wages on some interval). Likewise, if \(w(\hat{\tau}) = p_l\), then for \(\hat{\tau} < t < \hat{\tau}\), \(\dot{V}_s \leq \dot{V}_g\) (going backwards from \(V^-\)), it must be that \(V_s(\hat{\tau}) \geq V_g(\hat{\tau})\), strict if there is an interval of time in which wages differ. Hence \(V_s > V_g\) for \(t\) in the interval. Now, look at \(J_s\). Going backwards from \(\hat{\tau}\), both policies give a value \(V_0\) to the worker. However, at any time \(\dot{J}_s < \dot{J}_g > 0\), given that \(V_s > V_g\).
over the entire interval. Then, given that they end up at the same $J^-$, it must be that $J_s(0) > J_g(0)$. Since $\Pi_s(0) = J_s(0) - V_0 + V_u$, it follows that $\Pi_s(0) > \Pi_g(0)$. ■

This lemma shows that I only have to look at contracts that have wages at $c$, $p_l$ or $p_h$. The next proposition tells me that I can restrict this even further: the firm will never offer $p_l$ wages before it dismisses the low productivity worker, unless it offers wage $p_l$ without dismissing at all.

First, notice the following about maximizing the expected surplus under static (time-invariant) wages. With time invariant wages the expected surplus is maximized by the highest wage possible, subject to the firm’s participation constraint. I will index the values of such a policy with capital letters (L,H). Looking at equation (23), one can see that the surplus is diminishing in the distance between $V$ and $\Pi V$. A higher wage brings these closer together, naturally, because the worker will incorporate more of the value of the match when he gets an outside offer. For the case that firm does not fire the worker in a low-productivity match, the maximum static surplus has wage $p_l$ and solves

$$\Pi_V = \alpha p_h + \alpha \lambda \int_{V_L} \max\{\hat{V}, \Pi^*_V\} dF(\hat{V}) \frac{\delta + \lambda}{\lambda}$$ \hspace{1cm} (25)

where from (20), it follows that

$$V_L = p_l + \lambda \int_{V_L} \max\{\hat{V}, V_L\} dF(\hat{V}) \frac{\delta + \lambda}{\lambda}$$ \hspace{1cm} (26)

At this wage, firms make a profit over the workers in high-productivity matches:

$$\Pi_L = \alpha p_h \frac{\delta}{\delta + (1 - F(V_L))}$$

and every time a high productivity worker gets an outside offer $\hat{V} > V_L$, he does not take $\Pi_L$ into account.

On the other hand, I can look at the value under a $w = p_h$ policy. In this case, the firm will find it better to dismiss the $p_l$-worker. From (24), and looking in expected terms (in the equation below), one gets

$$\delta \Pi^*_h = p_h + \lambda \int_{\Pi^*_h} (\max\{\hat{V}, \Pi^*_H\} - \Pi^*_H) dF(\hat{V})$$ \hspace{1cm} (27)

$$\alpha \Pi^*_H = \alpha p_h + \alpha \lambda \int_{\Pi^*_H} \max\{\hat{V}, \Pi^*_H\} dF(\hat{V}) \frac{\lambda + \delta}{\lambda}$$ \hspace{1cm} (28)

Note that for $J_H = \Pi^*_H - V^u$, a firm with a low-productivity match will have dismissed its worker, whereas this worker still contributes to $J_L = \Pi^*_L - V^u$. However, for
the latter case matches with high productivity are mistakenly broken up, because the worker leaves, not taking into account the profit destroyed at the high-type firms. The trade-off therefore consists of match breakups due to shutting down low-productivity matches, and match breakups due to workers moving away from a highly-productive firm. This can be seen more accurately when rewriting the inequality for increasing efficiency with time-invariant wages, in ex ante terms

\[ \alpha J_H > J_L. \]

**Proposition 2** *In equilibrium, the firm breaks up a match with productivity \( p_i \) as soon as the wage \( w \) in the contract jumps up to \( p_i \).*

**Proof** See appendix \( \blacksquare \)

The intuition here is that when it is worthwhile to dismiss the worker in the low-productivity match, paying a wage \( p_l \) before dismissal effectively constitutes front-loading. The firm would be better off promising the surplus that falls to the worker after dismissal at a sooner time. If, on the other hand, wages at \( p_l \) would not constitute front-loading, this would contradict the optimality of the dismissal decision.

Now, let us look at the conditions under which the firm will offer \( p_h \), after some (still unspecified) time \( \hat{t} \). I find the conditions under which the firm prefers a jump straight to \( p_h \), as opposed to jumping to \( p_l \) without dismissals, and show that if the firm would prefer to \( p_l \) with dismissals as a first step, it still prefers a big step to \( p_h \) to a step to \( p_l \) without dismissals.

**Proposition 3** The firms will offer a contract with \( p_h \) from some time \( \hat{t} \) onwards if and only if the maximum joint surplus of offering \( p_h \) is larger than the maximum joint surplus of offering \( p_l \). If, moreover, the expected surplus \( \alpha J_H \) is large enough, it will offer \( p_l \) wages (after a possible dismissal) for some interval of time, before jumping to \( p_h \).

Firms will raise the wages higher than \( p_l \) if it is the expected joint surplus at constant wage \( p_h \), \( \alpha J_H \), is higher than the surplus at constant wage \( p_l \), \( J_L \). Moreover, when \( \alpha J_H < J_L \), but the difference is sufficiently small, the firm will still find it optimal to raise wages above \( p_l \). Note that latter implies that the joint surplus maximizing

\[ \alpha J_H > J_L. \]

\[ ^{16} \text{The latter part looks like something that would happen with discrete productivities, but would disappear when one goes to a continuum of productivities.} \]
policy is not necessarily a constant wage policy! However, even with these non-constant wage policies, the jump still occurs if and only if it is (constrained) efficient that workers in low productivity matches are fired. By constrained-efficient I mean, in this context, privately efficient subject to the limited commitment of participation. In other words, this takes into account that some surplus cannot be reached, because one of the parties would walk away from the match.

**Proof** I show that a contract with all wages $w \leq p_l$ is dominated by some contract with some wages at $p_h$, if and only if the joint surplus of a contract with $p_h$ is larger than a contract with $p_l$. From the previous analysis, we know that contracts without dismissals can only jump from $c$ to $p_l$. Now, take a step contract with dismissals which, moreover, jumps straight to $p_h$ at time $\tilde{t}$. Repeating the argumentation of lemma 1, while taking into account that some matches will be broken up, it can be showed that this contract dominates any contract without dismissal, if and only if the conditions are satisfied.

Note that at the time of the step, the (ex ante) value of the contract for the worker is $\alpha J_H + V^u > J_L + V^u$. For the firm, the contract that steps up to $p_h$ at $\tilde{t}$ has $\Pi(\tilde{t}) = \Pi_H(\tilde{t}) = 0$. I need to show that the ex ante gain for firms from posting a $p_h$-contract is larger, if and only if the conditions are satisfied.

To proceed towards this (and beyond), I have to introduce the recursive value functions for the surplus of three policies: $J_l, J_d, J_h$, where the lower-case letters denote that the contracts involve wages different than the productivity of the match. $J_l$ tracks the ex ante expected surplus of a policy that has $c$ up to some $\tilde{t}$, and then jumps up to $p_l$ without dismissal. $J_d$ follows the ex ante expected surplus of a policy that has $c$ till $\tilde{t}$ and then jumps up straight to $J_H$, with probability $\alpha$. Finally, $J_h$ follows the surplus, after dismissal, of a policy where the wage equals $p_l$ till some $\tilde{t}_2$, at which point it jumps up to $p_h$. I use ex ante expected surplus here because the firm is ultimately interested in the ex ante expected profit at time $t = 0$. $\dot{J}_l$ and $\dot{J}_d$ are used to trace back the expected surplus from $J_L$ or $\alpha J_H$, to $J_l(0)$.

$J_l$ is defined in the following way, taking $E[p] = \alpha p_h + (1 - \alpha)p_l$:

$$\dot{J}_l \equiv \delta \Pi_l - E[p] - \lambda \int_{V(\tilde{t})} (\dot{V} - \Pi_l) dF(\dot{V}),$$

where $V \leq V_L$ at all times. $b)$ the behavior of the surplus function of contract that jumps up at $\tilde{t}$ to $p_h$, thus involving dismissal, and providing identical value $V(0)$ to the worker. The joint surplus evolution before the jump is dictated by $J_d$, with a $d$ because
workers still have to be dismissed\textsuperscript{17}.

\[
\dot{J}_d = \delta \Pi V_d - \mathbb{E}[p] - \lambda \int_{V(t)} (\dot{V} - \Pi V_d)dF(\dot{V}),
\]

where $V$ can increase up to $\alpha \Pi H + (1 - \alpha) V^u$. And, finally, after the jump surplus is given by

\[
\dot{J}_h = \delta \Pi h - p_l - \lambda \int_{\Pi h} (\dot{V} - \Pi h)dF(\dot{V}),
\]

with $p_l$ either at $p_l$ or $p_h$. There is the familiar before-and-after condition

\[
J_d(\tilde{t}) = J_l = \alpha J_H.
\]

Now, let’s trace back the joint surplus over time.

**Lemma 2** If $J_d > J_l$ for some $\tau$, than $J_d > J_l$ for all $t \leq \tau$.

**Proof** We are concerned with the case that $V^0$ is the same for both policies. Note that both policies dictate $w = c$ for some interval. Note that $V_l(t) \leq V_d(t)$ for all on this interval. Then suppose that $J_d(\tau) > J_l(\tau)$.

\[
\begin{align*}
\dot{J}_d(\tau) &= (\delta + \lambda(1 - F(V_d)))J_d - \mathbb{E}[p] - \lambda \int_{V_d} \dot{V}dF(\dot{V}) \\
&< (\delta + \lambda(1 - F(V_l)))J_l - \mathbb{E}[p] - \lambda \int_{V_l} \dot{V}dF(\dot{V}) = \dot{J}_l(\tau);
\end{align*}
\]

Suppose that at some $\tau' J_d = J_l$. Then we need to have $\dot{J}_d > \dot{J}_l$ at some interval $\tau' < t < \tau$, while $J_d(t) = J_l(t)$. By the equation (30) about this is not possible. Contradiction. Since I have assumed the same initial condition $V_0$, and I can alternatively define $J_d$ as a function of $V$, not of $t$, where $J(V(t)) = J(t)$, over the time-interval where $w = c$.

Thus, if $J_d > J_l$ for any $t$, $V(t)$, then the dominating strategy for the firm is to offer a step contract with a step to $p_h$ and a breakup of the low productivity matches.

Now, if $\alpha J_H > J_L$, then $\alpha J_H = J_d(V^u + \alpha J_H) > J_L(V_L)$. Thus, in this case, the dismissal decision is straightforward. However, also if $\alpha J_H \leq J_L$ (but not too low), it might be that the optimal policy asks for a dismissal.

**Lemma 3** If $J_d(V^u + \alpha J_H) = \alpha J_H$ is close to $J_L$, than $\dot{J}_d < 0$.

\textsuperscript{17}Our previous notation $J^{ecv}$ already captured this, but because I want to distinguish three contracts: $p_l$-step, no dismissal contracts, indexed by $l$, $p_h$-step dismissal contracts, indexed ex ante by $d$, and after the dismissal by $h$. 

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\[ \dot{J} = \dot{J}_d - \dot{J}_l = 0 \]

Subtract \( \dot{J}_l \) from \( \dot{J}_d (\alpha J_H + V^u) \), to find \( \dot{J}_l < 0 \)

\[ \dot{J}_l = \dot{J}_d - \dot{J}_l \approx \delta \Pi V_d - \mathbb{E}[p] - \lambda \int_{\Pi V_d} (\hat{V} - \Pi V_d) dF(\hat{V}) - \delta \Pi V_d + \mathbb{E}[p] + \lambda \int_{V_H} (\hat{V} - \Pi V_d) dF(\hat{V}) \]

\[ = \int_{V_H} (\hat{V} - \Pi V_d) dF(\hat{V}) < 0, \quad (31) \]

where \( \Pi V_d = V^u + \alpha J_H \). As \( J_d(\check{t}) \) increases, moving further away from \( J_l \), \( \dot{J}_d(\check{t}) \) will increase, as \( d\dot{J}_d/dJ_d = \delta + \lambda (1 - F(J_d)) > 0 \).

Now, I want to establish when there is an interval of time that wage \( p_l \) is offered. Let us first establish the slopes of the surplus, just before \( p_H \) will be given to the worker, for as long as he chosen to remain at the current match.

**Lemma 4** For \( V \) in a neighborhood of \( V_H = J_H \), the optimal policy is wage \( p_l \) which jumps up to \( p_H \), rather than a wage \( c \) which jumps up to \( p_H \) if and only if \( J_d^{ss}(\alpha J_H) < \alpha J_H \).

**Proof** Define

\[ J_d^{ss} = \delta \Pi V_d^{ss} - \mathbb{E}[p] - \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V}) \]

The first step is to prove that \( J_d^{ss}(\alpha \Pi V_H) < \alpha J_H \), if and only if \( \dot{J}_d(\alpha \Pi V_H) > 0 \),

where \( \Pi V_H = \alpha J_H + V^u \). Take \( \dot{J}_d(\alpha \Pi V_H) \) and subtract \( \dot{J}_d^{ss} = 0 \) from it

\[ \dot{J}(\alpha \Pi V_H) = \dot{J}_d(\alpha \Pi V_H) - \dot{J}_d^{ss} \]

\[ = \delta \alpha \Pi V_H - \mathbb{E}[p] - \lambda \int_{\alpha \Pi V_H} (\hat{V} - \alpha \Pi V_H) dF(\hat{V}) \]

\[ - \delta \Pi V_d^{ss} + \mathbb{E}[p] + \lambda \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V}) \]

\[ = \delta (\alpha \Pi V_H - \Pi V_d^{ss}) - \lambda \int_{\check{V}_d^{ss}} (\hat{V} - \alpha \Pi V_H) dF(\hat{V}) \]

\[ + \lambda \int_{\Pi V_d^{ss}} (\hat{V} - \Pi V_d^{ss}) dF(\hat{V}) \]

\[ > 0, \text{ if and only if } \Pi V_d^{ss} < \alpha \Pi V_H \iff J_d^{ss} < \alpha J_H. \]

The intuition is clear here: if one is at a lower steady state \( (J_d^{ss}) \), and now suddenly a switch is instituted to the high steady state \( \alpha J_H \) at some future time, the value of \( J \)
will start to rise over time, i.e. $\dot{J}_d > 0$. Likewise, if the value is rising while before we were at steady state value $J_d^{ss}$, we must be moving a new $J$, higher than steady state $J_d^{ss}$.

The second and last element that is needed is that $\dot{J}_h = 0, \dot{J}_d > 0$ implies that at least for a neighborhood of $J_H$, it is better to provide value through $p_l$, than through wage $c$.

To show this, note that given $\dot{J}_h, \dot{V}_h$, and terminal condition $J_H = V_H$, one can implicitly define a function $J_h(V)$. Likewise, an implicit function $J_d(V) = J_d(\alpha V + V^u)$ can be found, from $\dot{J}_d, \dot{V}_h$. Now,

$$\frac{d^- (\alpha J_h(V_H))}{d(\alpha V)} < \frac{d^- (J_d(V_H))}{dV},$$

where the minus-sign denotes the right derivative at $J_H$ and $(\alpha J_H + V^u)$, implies that given a $V$ in some neighborhood close enough to $V = V_H$, it is better to choose the $J_h$ policy (with wages $p_l$) than the $J_d$ (with wages $c$, and dismissals still to come). But one can find

$$\frac{d^- (J_h(V_H))}{dV} = \frac{d^- (\alpha J_h)/dt}{d(\alpha V)/dt} = \frac{\alpha J_h}{\alpha V_h} = 0,$$

and

$$\frac{d^- (J_d(V_H))}{dV} = \frac{d^- J_d/\dot{V}_d}{d^- V/\dot{V}_d} = \frac{\dot{J}_d}{\dot{V}_d} > 0,$$

if and only if $J_d^{ss} < \alpha J_H$. Thus for some neighborhood of $V_H$ it is optimal to have wages at $p_l$ after a dismissal, if and only if this condition is satisfied.

Note that for the specification above a jump to $p_l$ and then to $p_h$ is also constrained efficient. Note that $J_L < J_d^{ss}(\alpha J_H) < \alpha J_L$, so the case directly above is a subset of the cases where constraint efficiency requires jumps to $p_h$, as derived before. It is just that $p_h$ is preceded by some period of $p_l$.

Thus, above I have derived the condition for a wage schedule with wages at $p_h$ at some time interval. It turns out that this jump will happen if and only if it is constrained privately-efficient to have a wage at $p_h$. The joint surplus-maximizing wage policy might call for a step contract itself (it is not necessarily a constant wage at one of the productivities, as one might have expected). The intuition for the latter contract is the following: the low productivity matches add a lot to the surplus, but the threat of outside offers that destroys the firm’s profit is also large. Under the $p_l$ regime, the firm cannot promise more than $V_L$, but under the $J_d$ regime, the firm can. In this case, the ex ante expected wage surplus $J_H$ is less than $J_l$, but the additional
profit cashed in while \( V \) went from \( V_L \) to \( \alpha V_H \) just compensates that. The second implication is that there might be some \( \alpha J_H < J_L \) that still call for a dismissal as the joint surplus maximizing policy.

All in all, I have found the following: if \( \alpha J_H > J_d^{ss} \), the contract has a step at \( p_l \), before it jumps up to \( j_h \). If \( J_L < \alpha J_H < J_d^{ss} \), and even for some values below \( V_L \), the optimal wage policy comes with one big step from \( c \) to \( p_h \). Finally, if \( \alpha J_H \ll J_L \), the optimal policy is a wage at \( c \), followed by a raise to \( p_l \), without dismissals. ■

**Remark 1** The case with the declining \( J_d \) function seems specific to the setup with discrete productivities.

Note that in terms of efficiency and dismissals, I get the result that firms will dismiss workers if and only if it is ex ante constrained efficient to do so. Although, the decision whether to eventually dismiss workers or not coincides with constrained efficiency, the firms delay the decision to to extract rents, so the timing of the dismissal is generally not ex ante constrained efficient.

Up to now, it was derived that the optimal contract is a step contract where the steps coincide with the productivities. The dismissals occur at the first step (if there is a second step, or a big jump straight to \( p_h \)). The crux of this is that the optimal contract can be characterized as a simple sequence of stepping times \( \{\tilde{t}_i\} \), where \( \tilde{t}_i = \tilde{t}_{i+1} \) is denoting a step of two productivities at once.

Thus, given the conditions are met for ending up at \( p_h \), and having an intermediate step at \( p_l \), the characterization of the stepping times comes from solving the following problem:

\[
\max_{\tilde{t}_1, \tilde{t}_2} \left( \int_0^{\tilde{t}_1} e^{-\delta t} \psi(t)(\alpha p_h + (1 - \alpha)p_l - c)dt + \int_{\tilde{t}_1}^{\tilde{t}_2} e^{-\delta t} \psi(t)(\alpha p_h - \alpha p_l)dt \right) \tag{32}
\]

subject to

\[
V_0 = \int_0^{\tilde{t}_1} e^{-\delta t} \psi(t) \left( c + \frac{\hat{V}}{\tilde{V}(t)} \right) dt + \alpha \int_{\tilde{t}_1}^{\tilde{t}_2} e^{-\delta t} \psi(t) \left( p_l + \frac{\hat{V}}{\tilde{V}(t)} \right) dt \\
+ \alpha e^{-\delta \tilde{t}_2} \psi(\tilde{t}_2) V_H + (1 - \alpha) e^{-\delta \tilde{t}_1} \psi(\tilde{t}_1) V_H
\]

and \( \hat{V}, \psi(t) \), derived analogous to (6), (4).

From this problem, the characterization of the stepping times, (given \( \alpha J_H > J_d^{ss} \)), is given by the following proposition.
Proposition 4 The conditions for the steps in a two-step contract are given by the following: going backwards in time, there is a step down from \( p_l \) to \( c \), at time \( \tilde{t}_2 - \tilde{t}_1 \) when

\[
\frac{\dot{\Pi}^{ecv}(\alpha \Pi_h, \alpha V_h + (1 - \alpha)V^u)}{\dot{V}^{ecv}(\alpha V_h + (1 - \alpha)V^u)} = \frac{\dot{\Pi}^{cv}(V_h, \Pi_h)}{\dot{V}^{cv}(V_h)}
\]

at some \( \Pi, V \), taking \( \dot{J}, \dot{V} \) backwards from \( V_H = J_H \).

Proof See appendix \( \blacksquare \)

Whereas the proposition above finds the general characterization of the stepping times, I am also interested in whether such a 'first' step to \( p_l \) occurs. One could think of cases in which it could be better to start offering wage \( p_l \) and firing the worker with a bad realization right away. A sufficient condition for this not to be the case (i.e. the wage contract will start at \( c \)), is given by the following:

\[
0 = (\delta + \lambda(1 - F(V)))\alpha \Pi - \alpha(p_h - p_l) > (\delta + \lambda(1 - F(\alpha V)))\alpha \Pi - \alpha(p_h - p_l) - p_l + c
\]

which leads to

\[
\frac{\lambda(F(V) - F(\alpha V))}{\delta + \lambda(1 - F(V))} < \frac{p_l - c}{\alpha(p_h - p_l)},
\]

where I have used that \( \dot{V} > 0 \) in both cases. This condition basically tells us that the \( \dot{\Pi}/\dot{V} \) lines will cross at some \( V \), thus cause a step down, from \( p_l \) to \( c \) (if one were going backward). This sufficient condition is derived from the fact that \( \dot{\Pi}^{ecv}(V) > 0 \), while \( \dot{\Pi}^{cv}(V) = 0 \), for the particular \( V \) in question. Then the step up will take place after that \( V \), while it is at the same time profitable to offer this \( V \geq V^0 \).

Thus, all in all, I have derived the following (by combining the previous lemmas and propositions):

**Theorem 1** Supposing that \( \alpha J_H > J^s_d \), and condition (34) is satisfied, the optimal contract consists of two steps, one to \( p_l \) (while firing) at \( \tilde{t}_1 \), and another one to \( p_h \) at \( \tilde{t}_2 \) (without dismissal), where \( \tilde{t}_2 > \tilde{t}_1 \).

The conditions basically require that competition of other firms is not so strong that a worker who is unsure whether he will be fired will opt for an outside offer within no time. In this case, the firm will have to tell him right away by a wage of at least \( p_l \). On the other hand, it also cannot be the case that the worker is almost completely sure that he is a high type, in which case he will almost completely disregard to dismissal risk. In this case, the firm might find extreme backloading the optimal policy. And
Figure 2: An optimal two-step contract

again, the third, perhaps simplest condition is that it must be in the ex ante joint interest of the match to fire the low productivity worker. If the cost of dismissal is too high, for example when the productivities differ not by much, then it is best to never jump above $p_l$, and keep the lower productivity worker around.

5 Discussion

5.1 Discussion of the Optimal Contract

In the previous section, I have derived the optimal contract. I have shown that if the right conditions are satisfied, the optimal contract comes in two steps, with the first step below the high productivity level, and the low-productivity worker would be fired at the first step.

In this case, the firm chooses not to reveal information right away. It prefers to keep the worker in the low-productivity match around for some time. Because the worker in the low match does not know what type he is, he is not leaving faster than a worker in a high-productivity match. Thus, initially, the firm prefers to pay the lowest wages possible, exploiting both the good and bad productivity employees (but, in a sense, the bad worker more, because the firm will never pay the worker in a low-productivity match more than $c$).

However, if the firm would engage in extreme backloading (where the wage would jump directly from $c$ to $p_h$), the worker would discount the promised wage increase significantly more than in the gradual case, because of the uncertainty about the worker’s retention prospect. Even more so, it would, up until the moment of the jump, also lead good workers to accept outside offers which they would have been better off rejecting,
and the low-productivity workers would reject offers they would have been better off taking. This is costly for the worker, it decreases the value of being in the relationship and, because this leads to decreased participation of the worker, it is costly for the firm too.

Therefore, it is optimal for the firm to commit to an early wage jump to a level smaller than \( p_h \). If the worker is still there after the jump, he can infer that he must be in a good match. In the two-productivity case, this means that he is certain that the firm will keep him, and eventually will pay him \( p_h \). As a result, the worker in a good match is a lot less willing to leave the firm following the jump, while the firm is still able to make a profit from his service over some time interval.

The model captures the spirit of up-or-out contracts, with \( p_h \) and \( p_l \) interpreted as cut-off productivities. The firm commits in advance to a wage schedule, and a ‘tenure’ time with a wage increase. Up until that time, the firm keeps employing the low and the high-productivity worker, as the low-productivity worker is still useful at the given wage. (Think e.g. of law firms or accounting firms, where lawyers or accountants who are not that good are still productive, but just not good enough to be made partner, or of assistant professors prior to the tenure decision.) At the time of the ‘tenure decision’, the match quality is revealed (through the pay raise), even though the wage is not hiked up to maximum possible level. A good worker will now be tied more to the firm; he will no longer be easily tempted to leave for other firms.

As noted before, the contract is set in a second-best world, as I assume that the worker cannot see what he is worth to the firm, and the firm cannot see what offers the worker is getting. I assume that the firm does not have a cost-effective mechanism at its disposal that can make the worker truthfully reveal his outside offer, so this non-observability has a bite. As a result, the wage contract tries to approach the first-best by rewarding duration, but in a way that also involves mitigating the adverse impact of the participation decisions of the firm. Thus, wages will grow at an early stage, and the participation risk of the firm, which is based on a persistent factor, is resolved then. This means that whenever the worker becomes more certain about the participation of the firm, the firm can engage in more cost-effective backloading.

All in all, there are a lot of strategic forces at work that shape the growth of wages over tenure. If, on the other hand, everything were observable, one would be in a first best world and contracts would take a simple form: the firm could take all the surplus until the worker got an outside offer, and then simply match whatever the competitor
wants to pay. This wage would be paid until a lower production shock might force the wage downward while preserving the match. In the case with productivity shocks i.i.d. across firms and time, matches only break up when a firm making the outside offer has a higher productivity than the firm the worker is currently in, or the productivity shock is so bad that a worker who would be paid that productivity level would prefer to be unemployed instead. This is the case that Postel-Vinay and Turon (2006) study. With firms that are subject to the same distribution of shocks, this leaves out not only the ex ante concern about participation, but also the interdependence of participation decisions. In my model, it is the interdependence of participation decisions that is a major cause and determinant of the wage growth pattern.

5.2 Future Research

An important assumption in the paper is that the firm commits to a ‘simple’ wage-tenure contract: it specifies a single wage for each period that the worker is employed at the firm. Many firms have standardized wage contracts, or a standardized tenure track, so this assumption is not without realism. Moreover, this assumption allows us to make the trade-off in terms of participation very clear-cut: a wage increase over $p_l$ would result in the dismissal of the $p_l$ worker. Not every contract has to be so stark. I could investigate alternative behavior during the match. Two issues are particularly interesting: what would the contract look like if the firm could keep the worker in the low productivity match, instead of firing him? And, at what time would the firm want to tell a worker he is good if they could renegotiate the contract during the relationship? The general framework I constructed above allows these questions to be addressed as well. In that sense, the full commitment contract is the first step in this agenda, with more to come.

Another interesting dimension that can be relaxed is specification of uncertainty on the firm’s side. In the model, the uncertainty resolves at the beginning of the match, the worker just doesn’t know about the outcome. Wages are informative because the shock is persistent, but nothing requires persistence to take a once-and-for-all form. I can generalize this setup to include uncertainty that realizes during the match. As long as there is some degree of persistence of the productivity shock, wages will be informative. An interesting application would be the response of wage schedules to the amount of uncertainty associated with working for a firm.\textsuperscript{18}

\textsuperscript{18}An interesting observation in this respect is that Life-cycle wage profiles have flattened over the
So far, I have taken the outside offers to be exogenous. However, it is possible to solve for the market equilibrium along the lines of Stevens (2004) and Burdett and Coles (2003). This is important as the distribution of outside offers is not invariant to, e.g., the distribution of productivities in matches or the amount of uncertainty in the economy. Hence, wages do not only respond to the uncertainty through the worker’s belief about match quality, but also through the outside offers that the firms have to compete with. Studying the equilibrium responses in the market to changes in aggregate uncertainty might gain us additional insights about the U.S. labor market.\textsuperscript{19}

In its current version, the model speaks to \textit{qualitative} aspects of the data. Needless to say, a less stylized version of the model could aim at quantitative analysis as well. The list of topics to be addressed, apart from those mentioned above, also include other observations. For example, workers with higher tenure that are dismissed often incur a larger wage loss. The model could also address the pattern of job separations over the job relationship, which is hump-shaped with a slowly declining tail (one that is declining too slow to be adequately explained by pure learning alone (Van der Ende and Teulings 2002), and the evolution of layoff hazards over job relationships as well.

6 Conclusion

In the paper, I have proposed a theory of long-term contracting that takes the participation uncertainty of both worker and firm seriously. The optimal contract was characterized, which trades off instantaneous profit with the need to retain the firm’s workers, but in particular the workers in high-productivity matches. The resulting wage-tenure profile exhibits \textit{gradual} growth over the course of the employment relationship, driven by (and responsive to) the bilateral nature of uncertainty. A direct implication of this model is that the uncertainty a worker attaches to his employment

last decades (Kambourov and Manovskii 2004,2005) (Marcotte (1998) has found that the return to tenure has decreased over time, while at the same time, volatility of firm performance and profit has gone up (Comin and Phillipon (2005)). A theory set in the framework of bilateral uncertainty could provide a link between wage profiles and firm volatility, and could have something to say about turnover patterns as well (e.g. decreased tenure on jobs, not only because of shocks but because of changes in the labor market equilibrium). A further interesting dimension is that the patterns of job insecurity have changed as well, with the risk of layoffs shifting towards more senior workers; the model presented here would allow an investigation of whether changes in wage profiles could be accounted for by these shifts.

\textsuperscript{19}This part of the model is currently work in progress.
affects the optimal wage profile that the employer wants to give him over the course of the relationship. The model can be readily extended to study this effect in a market equilibrium as well. It can thus speak to important features of the data, such as wage growth with tenure, the decline of turnover and dismissals over an employment relationship, and the link between uncertainty properties of the economy and labor market outcomes.

As a final note, it is interesting to see that this alternative theory about wage growth has very different implications for allocative efficiency, than for example, human capital. In this model firms try, by backloading wages, to monopolize the worker over time. However, how effectively they can do this depends on the certainty about continued employment at the firm. A dynamic, or perhaps ‘turbulent’ economy might therefore have an additional benefit: it could induce firms to flatten wage schedules, which would make workers more free to move from place to place.

Appendix

A Proof of Propositions

Derivation of Equation (12) One can rewrite

\[ \delta V^u = b + \lambda \int_{V_u} (\hat{V} - V^u) dF(\hat{V}) \]

as

\[ \delta V^u = b + \lambda (1 - F(\hat{V})) V^u + \lambda \int_{V^u} (\hat{V} - V^u) dF(\hat{V}). \]

Then, it follows that

\[ V^u = \int_0^\infty (e^{-\delta t} \psi(t)) \left( b + \lambda \int_{V^u} \hat{V} dF(\hat{V}) + \int_{V^u} (\hat{V} - V^u) dF(\hat{V}) \right) dt. \]

Now use this formulation of \( V^u \) in \( \Pi + V - V^u \), to get equation (12).

**Proposition 1** An optimal contract will have wages that are weakly increasing over time; the optimal quit decision therefore occurs at \( w = p \). (Or, if wages jump up, at the first time that \( w \geq p \).)

**Proof of the case with a continuous \( G(p) \)** Let us first consider the case that \( G(p) \) is continuous. Suppose that there is a decrease in wages in the optimal contract. I want to construct a non-decreasing wage schedule that is preferred by both parties, strict by at least one of them.
Concretely, by a decrease in wages I mean that I can find \((t_1, t_2)\) with \(t_1 < t_2\), such that there exists a interval of times between \(t_{01}\) and \(t_{02}\), with \(t_{01} < t_{02} < t_1\), such that 

\[
\forall t_0 \in (t_{01}, t_{02}), w(t_0) > w(t), \forall t \in ((t_1, t_2))
\]

For this proof, it is important to understand when dismissal (i.e. break-ups induced by the firm) take place. Hence the following lemma

**Lemma 5** A worker is never dismissed during a time interval that where are decreasing. Moreover, if wages increase continuously, then a worker is fired only at \(w = p\). (Or, if wages jump up, at a time \(\hat{t}\), where \(w(t) \geq p\), whereas there was an interval \(T = [\hat{t}, \tilde{t}]\), such that \(w(t) \geq p \forall t \in T\))

**Proof** Denoting \((t_1, t_2)\) to be an interval in which wages fall, it must be that there exists an time-interval around and including \((t_1, t_2)\) in which no dismissals occur. This follows from the simple fact that a firm who was not willing to fire workers at \(t_1\), will for sure receive an positive, or less negative instantaneous flow of profits over the \((t_1, t_2)\) interval.

In particular, At \(t_1\), \(\Pi(t_1, p) \geq 0\). If \(w(t) > p\), then expected profit \(\Pi(t, p)\) increases during the decrease in wages, by the fact that \(\Pi(t_1, p) \geq 0\) given the optimal decision \(t(p)\). A layoff time, \(t(p)\), during the time-interval of the decreasing wages interval contradict the greater than zero profit at the beginning. In words, the firm is paying \(w(t) > p\) as an investment, to get a larger profit later. Since it was optimal to start investing, it is optimal to continue. Therefore, the firm should hang on to the worker. If \(p > w\), then the firm is making profit, and there can be no reason to fire the worker.

It is straightforward to see that dismissals of a match that drew \(p\), occur only at \(w(\hat{t}) = p\), or at \(w(\tilde{t}) > p\) only if \(\lim_{t \uparrow \tilde{t}} w(t) < p\).\(^{20}\) By the above reasoning, they cannot occur in a decreasing wage-interval. Thus, wages must be increasing (weakly) at the time of any dismissal. Then, if they would occur sooner, a profit could be made by postponing, if they occur later, a profit could be made by dismissing earlier. 

There are four cases: there is an interval of decreasing wages before anyone is fired (case 1); secondly, there is a interval of decreasing wages between the dismissal of two types (case 2). And finally, workers are fired initially, and wages decrease subsequently without any dismissals (case 3); and finally, there is a decreasing wage interval but there no layoffs at any time.

I prove the proposition by constructing an alternative (weakly) increasing wage schedule that yields at least as much profit to the firm, and at least as much value to the firm, for these three cases. The basic idea is the following: if one redistributes the wages in such a way that a) one backload wages as much as possible, without b) violating the firm’s break-up condition, while c) keeping the initial values constant for both worker and firm constant, given unchanged break-up decisions of workers, then it follows that when one allows workers to change their breakup decisions given the new wage schedule, both workers and firms are better off. Workers by revealed preference,\(^{20}\)

\(^{20}\)I focus on wage-tenure contracts that are a.e. continuous, and a.e. differentiable in this paper. I show that they are increasing and bounded, so this property is natural.
and firms because workers’ revealed preference is to actually stay longer with the match. Since each productivity was profitable, and the wages are never so high as to induce a new breakup, this is better for the firm too.

□ Case 1  Wages decrease on some interval, before $t_f$, where the first guy is fired. Note first that it must be that $t_f > c$, where $c$ is the minimum wage the firm can pay. Let $\Pi(p, t_f)$ be defined as

\[
\Pi(p, t_f) \equiv \int_0^{t_f} \psi(t)(p - w(t))dt,
\]

and $\bar{V}_{t_f}$ as

\[
\bar{V}_{t_f} = \int_0^{t_f} \psi(t)(w(t) - b - \lambda \int_{V_u} \bar{V} dF(V))dt
\]

Note that I have defined $\bar{V}$ here as the value for the worker above unemployment.

Now, given that I keep $\bar{V}(t)$ constant (given that the break-up decisions are also held constant!), to constraints for the new, backloaded wage schedule $w^b(t)$ are

1. $c \leq w^b(t) \leq p \forall t, 0 \leq t \leq t_f$
2. $\int_0^{t_f} \psi(t)w(t)dt = \int_0^{t_f} \psi(t)w^b(t)dt$

Find $t^{bs}$ such that $t^{bs} = \{t | \int_0^{t_f} \psi(t)w(t)dt = \int_0^{t_f} \psi(t)w^b(t)dt \}$. This $t^{bs}$ exists, otherwise it would have been optimal to fire $p$ at $t = 0$.

Let the worker now choose optimally, given the new wage schedule. He is always able to copy his old acceptance schedule of outside options, and thus always able to obtain his old value. Any new choice is therefore an improvement for him. For the firm, note that the new wage schedule implies that current value $\bar{V}(t)$ for worker is higher than before, at any $t > 0$. This means that the new $\psi^{bs}(t)$, given the new optimal responses of workers is always higher than before, i.e.

\[
\Pi^{bs}(p, t_f) \equiv \int_0^{t_f} \psi^{bs}(t)(p - w(t))dt > \Pi(p, t_f) \equiv \int_0^{t_f} \psi(t)(p - w(t))dt
\]

This will in fact hold for any $p$, and therefore also for the ex ante expected profit.

Hence, I have constructed a wage schedule that (pareto) dominates the old schedule with the decrease, and is strictly better for the firm.

□ Case 2  Take an interval between two dismissals, with an sub-interval of decreasing wages in between. I will show that a constant wage will do better. The firm is indifferent between firing a worker at the beginning and at the end of the $(t_1, t_2]$ or $[t_1, t_2)$ interval, when this worker has a productivity $p$, associated with either the maximum of productivities fired before $t_1$ (in case $(t_1, t_2)$), or the minimum of the productivities fired after the time interval (in case $[t_1, t_2)$). This means that

\[
\Pi(p) \equiv \int_{t_1}^{t_2} \psi(t)(p - w(t))dt = 0
\]
Now, let’s replace the \((t_1, t_2)\)-interval of original wage-tenure contract, by a constant wage \(w = p\). This will yield zero profit to the firm, if productivity is \(p\). Along the same lines as above, I can show that the worker will do better by this contract, and decide in strictly more cases to stay with the firm. Moreover, any firm with \(p' > p\) will do strictly(!) better.

**□ Case 3** There is a decreasing wage schedule, strictly decreasing over some time, and all dismissals occur at the beginning. Again, look at the supremum of the productivities fired at the beginning, \(p\). At this productivity is the firm is indifferent. Consider the alternative of a constant wage, \(p\). Again, going through the same reasoning as above, this will give strictly more profit to firms, and weakly more value to workers.

**□ Case 4** Follows case 1 close, but now over an unbounded time interval. Take \(p\) to be the lowest productivity at which workers are fired. Clearly, at no point did profit for this productivity become negative. This means that at no point did the firm promise an amount to the worker that it could not pay unless it would raise wages above \(p\). Hence, the problem reduces to finding \(\hat{t}\) before which \(w(t) = c\), after which \(w(t) = p\), while providing the same value (holding constant match-breakup decisions of workers). This \(\hat{t}\) exists, and by revealed preferences workers will do better, and by strictly increased match survival firms will do better.

Putting all pieces together: there cannot optimally be a strictly decreasing part of a wage-tenure contract. Hence, the first time \(w = p\), or \(w\) jumps over \(p\), the firm will disband matches with productivity \(p\). Note that the strictness was derived under the assumption of a continuous distribution of \(F(V)\) with full support. The proof can straightforwardly be adapted for the case without full support (in which these preferences will be weak), and the statement is that there exists an increasing contract that is not dominated by any other contract.

Note that the last case basically provides an alternative proof of the (weak) optimality of step-functions as the optimal wage-tenure relation in the standard case in Stevens (2004). The proof is perhaps a bit longer (but mainly since one have to worry about firm-induced dismissals here), but it has a very intuitive ring to it.

**Proof of the case with a discrete r.v. \(\sim G(p)\)** The different cases basically follow the same line; hence I pick one case to illustrate the difference. The other cases follow along the same lines. Let us focus on case 2 because continuity has most bite there.

Take again the time interval between two dismissals \((t_1, t_2)\), with a decreasing wage during some sub-interval of time. Now, one does not have the indifference in profit. In fact, when \(p\) is fired, then productivity right above it, \(p'\) still makes positive profit. However, I can once again construct a step contract that gives equal profits, given constant break-up decisions, the firm and the worker.

Note that I have always constructed flat or increasing profiles, and therefore the previously optimal firing decisions stay optimal. If instead one would make very large steps (first down to
c, then up to the firing wage, this would also be a dominating contract, given the firing decision constant. However, the firing decision would adjust, and the contract with the changed firing decision would once again be dominated by an increasing contract. □

There is one cautionary note: one can construct non-generic examples where there is exact indifference between the surplus maximizing policy with and without dismissals. In this case there are nonmonotone contracts that are optimal too. However, there is always an monotone contract that achieves the values.

**Proposition 2** In equilibrium, the firm breaks up a match with productivity \( p_i \) as soon as the wage \( w \) in the contract jumps up to \( p_i \). Also, it is never optimal to break up matches with a productivity \( p_i \) at two or more different times.

**Proof** In principle there could be four cases of continuation values \((\Pi^-, V^-, J^-)\) at time of dismissal (incorporating the expectation of being dismissed), in relation to the values a match can achieve in expectation, without dismissals \((\Pi_L, V_L, J_L)\). The four cases are: 1) \( J^- > J_L, V^- > V_L \), 2) \( J^- < J_L, V^- > V_L \), 3) \( J^- > J_L, V^- < V_L \), and 4) \( J^- < J_L, V^- < V_L \). I want to show that in none of these cases is a dismissal preceded by a period of \( p_l \). I can restrict ourselves to contracts to offer either \( p_l \) or \( c \).

\( \square \) Case 1 These case share that \( V^- > V_L \). I will show that \( w = c \) is the best policy. Call \( w_s, V_s, J_s \) the wages and values of this policy, index by \( g \) some other policy. Then

\[
w_s(t) \leq w_g(t) \Rightarrow V_s(t) > V_g(t)
\]

by the same reasoning as in lemma 1. Note that as result of \( w_g > w_s \) during some interval, the dismissal time for each policy is different. But at the jump time \( \tilde{t}_s \), it must be the case that \( V_g(\tilde{t}_s) < V^- \), and \( J_g(\tilde{t}_s) < J^- \). As a result, take \( \dot{J}_s, \dot{J}_g \), and find that \( J_s(0) > J_g(0) \).

\( \square \) Case 2 Take the \( w_s = c \) policy, then, suppose \( V_0 = V_L \). Find the step time, and find \( J_s(0) \). If \( J_s(0) > J_L \), the optimal policy is to have a wage equal \( c \) till \( \tilde{t} \), at which one has \( J^-, V^- \). If \( J_s(0) < J_L \) (at \( V_L \)), the optimal policy is not to jump up with dismissal, instead it is to jump up to \( p_l \) without dismissals. At \( J_s(0) = J^- \), then it does not matter. In this case, the firm could offer a wage \( p_l \) before dropping back to \( c \). This is a non-generic case.

For case 3 and 4, I invoke a little lemma,

**Lemma 6** If the values at dismissal are \( V^-, J^- \), there always exists a policy that gives values \( V'^- > V^- \), and still achieve a surplus \( J'^- > J^- \).

**Proof** Look at (27). Call \( J_H(V) \) the surplus when the worker has a time-invariant value \( V \). Observe that \( J_H(V) \) is nondecreasing in \( V \) until \( V = J_H \). □
\[ \text{Case 3} \quad J^- > J_L, V^- < V_L \] with the appropriate change in policy after dismissal, from lemma 1, I can disprove the following: suppose it were optimal to have \( w = p_t \) for some time. Then, I would have \( V \) decreasing over some interval. Call \( \hat{V} \) the highest \( V \) gets. Since \( J^- > J_L \), \( J \) is increasing over the interval that \( V \) is decreasing. Moreover, the rate of the increase \( \dot{J} \) is faster than \( \hat{V} \) (\( \Pi > 0 \) as \( \Pi^- > \Pi_L \), and \( V < V_L \)). By the above lemma, one could also increase value at dismissal \( V^- \) to \( \hat{V} \), without changing \( J^- \) (even improve it possibly). Hence profit falls by less under the last alternative, as under the decreasing \( V \) policy. So no time of \( p_t \) wages is offered.

\[ \text{Case 4} \quad J^- < J_L, V^- < V_L. \] Along the same lines as case 3, I can show that one needs to be on the upward-sloping part of the \( V \) profile. Trace back the surplus from the \( J_L, V_L \), with \( w = c \) to \( V_0 = V^- \). If \( J(0) < J^- \), the optimal policy is \( c \) till \( V = V^- \), then dismiss. If \( J(0) > J^- \), then the optimal contract would be a non-dismissal contract that jumps up to \( p_t \), or a contract with a different \( J^-, V^- \), such that \( J(0) < J^- \).

**Proposition 4** The conditions for the steps in a two-step contract are given by the following: going backwards in time, there is a step down from \( p_t \) to \( c \), at time \( t_2 - t_1 \) when

\[
\frac{\Pi_{cv}(\alpha \Pi_h, \alpha V_h + (1 - \alpha)V_u)}{V_{cv}(\alpha V_h + (1 - \alpha)V_u)} = \frac{\Pi_{cv}(V_h, \Pi_h)}{V_{cv}(V_h)}
\]

at some \( \Pi, V \), taking \( \dot{J}, \dot{V} \) backwards from \( V_H = J_H \).

**Proof** Let us concentrate on \( V \) first. The problem should hold this constant. Split (32) into two parts:

\[
V_0 = \int_0^{t_1} e^{-\delta t} \psi(t) \left( w + \int_{V_0}^t \dot{V} dF \dot{V} \right) dt + e^{-\delta t_1} \psi(t) V_1^-,
\]

where \( V_1^- \) is the expected continuation value in the second stage. Likewise,

\[
V_1^- = \alpha \int_0^{t_2-t_1} e^{-\delta(t-t_1)} \psi(t - t_1) \left( w + \int_{V}^t \dot{V} dF \dot{V} \right) dt + e^{-\delta(t_2-t_1)} \psi(t) V_1 + (1 - \alpha)V_u.
\]

A change in \( t_1 \) causes an offsetting change in \( t_2 \), to keep \( V_0 \) constant. I solve this in two steps.

First, a change in \( t_1 \) leads to a change in \( V_1^- \). Then I calculate the change in \( t_2 \) to compensate for the change in \( V_1^- \).

\[
\frac{dV_1}{dt_1} = (w + \lambda \int_{V(t_1)}^\infty \dot{V} dF(\dot{V})) - (\delta + \lambda (1 - F(\dot{V}(t_1))) V_1.
\]

Notice that, since we have kept \( V_0 \) constant, we also have not changed any current value \( \dot{V}(t) \), notice that it equals \( \dot{V}(t_1) \) (which is not that surprising). The same trick can be repeated, for
the second part, but now we go back from $V_2$; in other words, we keep $V_2$ and the time that we get there constant. So again, $\hat{V}(t)$ is unchanged. For that purpose, let time go backwards:

$$V_2 = e^{\delta t_2} e^{\int_{t_2}^{t_1} \lambda(1-F(\hat{V}(\varsigma)))d\varsigma} V_1 - \int_{0}^{t_2} e^{\delta t} e^{\int_{t}^{t_2} \lambda(1-F(\hat{V}(\varsigma)))d\varsigma} \left( p_l + \lambda \int_{\hat{V}(t)} \hat{V} dF(\hat{V}) \right) dt$$

Then, $dV_1/dt_2$ yields the following:

$$\frac{dV_1^+}{dt_2} = (\delta + \lambda(1-F(V_1^+)))V_1^+ - p_l - \lambda \int_{V_1^+(t)} \hat{V} dF(\hat{V})$$

The $t_2$ here is the difference between $t_1$ and $t_2$ before (because we relabeled the time). Thus, we have derived the following:

$$\frac{dt_2}{dt_1} = -\frac{\hat{V}(\alpha V + (1-\alpha)V^u)}{\alpha \hat{V}_h(V)}$$

We can repeat the same exercise for $\Pi_l, \Pi_h$:

$$\frac{dt_2}{dt_1} = -\frac{\hat{\Pi}(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\alpha \hat{\Pi}_h(\Pi, V)}$$

Now, optimization would require that we keep increasing $t_1$ as long as $0 > \frac{dt_2}{dt_1} \bigg|_{\Pi} > \frac{dt_2}{dt_1} \bigg|_{V}$. As long as this is the case, the change that would keep profit equal makes the worker better off; and a change that makes the worker at least as well off makes the firm better off. So, it is good to increase $t_1$. Likewise, we could rewrite the problem in terms of the distance between $t_2$ and $t_1$. (It can be showed that the curves can only cross once.) Going backwards from $J_H, V_H$ (which is easier to do, to keep track of $\Pi$ and $V$), we would keep increasing the distance between $t_2$ and $t_1$ as long as $\frac{dt_2}{dt_1} \bigg|_{\Pi} < \frac{dt_2}{dt_1} \bigg|_{V}$. The switching point occurs when $\frac{dt_2}{dt_1} \bigg|_{\Pi} = \frac{dt_2}{dt_1} \bigg|_{V}$, or when

$$\frac{\hat{V}(\alpha V + (1-\alpha)V^u)}{\alpha \hat{V}_h(V)} = \frac{\hat{\Pi}(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\alpha \hat{\Pi}_h(\Pi, V)},$$

or

$$\frac{\alpha \hat{\Pi}_h(\Pi, V)}{\alpha \hat{V}_h(V)} = \frac{\hat{\Pi}(\alpha \Pi, \alpha V + (1-\alpha)V^u)}{\hat{V}(\alpha V + (1-\alpha)V^u)}.$$
References


