The Durable Information Monopolist

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Abstract

This paper develops a simple unified model of dynamic pure moral hazard rents in dynamic games and finance. We assume that an impatient individual has a unique competitive informational edge that would vanish were his actions perfectly observed. We start with a large insider who trades with the market camouflaged by noise traders (a two state version of Kyle, 1985). We then show that bounding his net order flow by one formally yields a repeated zero value game of incomplete and imperfect information — as arises in war or competitions. The informed player in this game has a secret preferred action, and the market response is then a mixed strategy and not a price.

The paper prices the value of information in these dynamic strategic contexts — namely, the ex ante expected profits of the information monopoly. If the insider can further deceive the market before exploiting his position, then the log-likelihood ratio of beliefs is the best measure of market deception. We then introduce and price the retrospective strategic *value of disinformation*. Choosing different actions in the states probabilistically erodes the information edge. We prove that the *information burn rate* (action differential) is highest when the market is most deceived, but exhibits diminishing returns: It explodes as the square root of the log likelihood ratio. We find that information is exploited more slowly when actions are more observable or the insider is more patient. In the limit of perfectly observable actions, a dynamic "No Trade Theorem" arises.

The market response depends on the nature of the information monopoly. The market price is an unbiased probability with unbounded burn rates. We then show that bounding the net order flow inflates prices when the market is optimistic and deflates them when pessimistic. As a result, the market rationally expects extreme prices to mean revert. In the repeated game, the mixed strategy responses of the uninformed player are analogously biased.

While information monopolies are ephemeral, we find that they never fully expire. The expected time until earning any fraction of the value is an increasing and concave function of the fraction. We find that "time is money" approximately, but that a longer monopoly need not be a stronger one.

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1 Introduction

In wartime, truth is so precious that she should always be attended by a bodyguard of lies. — Winston Churchill (to Stalin in Tehran, 1943)

Just before November 14, 1940, Winston Churchill faced a horrible dilemma. Just having cracked the German enigma code, he knew that the Luftwaffe planned to attack Coventry.[1] If Churchill saved the city and hundreds of lives, he would surely reveal his information source to the Germans, and grievously damage the war effort. Churchill did nothing, and Coventry was annihilated. This paper explores this "use it and lose it" feature of information exploitation in dynamic competitive strategic situations.

Churchill was faced with an all-or-nothing decision to defend the city, and the timing was given. We explore an informational asset that can be partially exploited at an endogenous timing. Separate literatures in finance and dynamic game theory have seen how an informational edge that is worthless with perfectly observed actions can be slowly eroded when actions are partially observed. We explore how fast these pure moral hazard rents are optimally burned, how valuable is the information, and how the market responds to this informational asymmetry. Our setting is simple enough that it can be fully solved, but rich enough that it subsumes both insider trading and repeated competitive games of incomplete information. We flesh out common insights.

Consider the 1815 Battle of Waterloo. Victory over Napoleon would be great news for Britain and its markets; defeat would surely be bad news. It was understood that Nathan Rothschild would know the outcome of the battle in advance of anyone else. Upon learning of the victory, Rothschild immediately allowed his known agents to sell consuls to signal a loss, while his unknown agents bought consuls at a tremendous discount later on. Rothschild's information monopoly lasted from June 20 until the next night before Wellington's messengers arrived. Rothschild entered the night a wealthy man, and therefore able to take tremendously long positions, exploiting his edge. He emerged vastly wealthier, having made a fortune before the public learned of Wellington's victory.[2] Our theory will suggest how Rothschild would trade most

[1]This, at least, is a premise of the current London play, "One Night In November", as well as Winterbotham's 1974 book "The Ultra Secret" (London, Weidenfeld and Nicolson).

[2]This possibly apocryphal event is described in www.thetruthseeker.co.uk/article.asp?ID=840. It surely inspired the climactic trading scene in the Academy Award winning movie "Wall Street".

intensely when the market was most deceived, and how to value his potential gains.

For our dynamic strategic thrust, we turn to Operation Fortitude in World War II. These were Allied activities to convince the Germans that the D-Day invasion would not occur in Normandy. This entailed diverting allied efforts away from Normandy, prepping invasions in Norway and the Pas de Calais that would not occur. Unlike the last example, the moral hazard rent owes to partially unobserved actions rather than noise trades. Also, the subterfuge efforts arguably came from a fixed budget. We will explore implication of this constraint, such as how it skewed the German response, and how this response is formally like a market price on the Normandy invasion choice. Our analysis here generally captures a range of competitive informational monopolies.

Let us start with a large insider who seeks to unload his position in an asset that he alone knows is overvalued, camouflaged only by noise traders (Kyle 1985). He trades in continuous time with competitive price-setting market makers at an unbounded intensity. The market response in this setting is a price, chosen to minimize the expected market losses from insider trading. The analysis of this problem is an optimal stochastic control exercise by Player 1. But unlimited exploitation of an informational upper hand is quite rare, since most traders can only take limited positions. Doing away with this flexibility, we next bound the absolute net order flow (buys minus sells) by unity, as is implicit in standard unit trade bid-ask models. The optimal stochastic control exercise also requires that we determine the regimes when traders are at their position limits.

This trade restriction opens the door to a much wider class of settings with a pure moral hazard rents. We re-interpret the net order flow as the difference of mixing chances in a continuous time binary action zero sum game of imperfect monitoring. Specifically, we assume that an informed player lives potentially forever. He might face a sequence of uninformed players who cannot see their predecessors' payoffs. Or — as in the D-Day example — he might play a single infinite-lived uninformed opponent, who sees payoffs when the world suddenly and unexpectedly ends. In either case, the market response is no longer a price but a mixed strategy.

We organize our investigation along four lines of inquiry: What is the *ex ante* and *ex post* value of the information monopoly? How fast is the information stock burned? What is the market response to this monopoly? How long does this monopoly last?

1. Values. The dynamic strategic value of information asks how much Rothschild should be willing to pay for his information network. We show that this prospective measure is a concave function of the market beliefs. But more central to our paper is the *value of disinformation.* This *ex post* value of the monopoly is Rothschild's expected payoff once he knows that Napoleon has lost. Considering the losses that Rothschild took in his initial market-deceiving trades, we argue that the log-likelihood ratio of the beliefs is a good measure of such deception. So if the market's initial odds on victory rise from 1:1 to 4:1 then his information monopoly is worth twice as much.

2. Information Burn Rates. Opposite actions are myopically optimal in the different states of world. But our informational monopolist also cares for his future, and shades away from a myopic course of action. Insofar as the subterfuge is imperfect, he erodes his informational edge. Barring price effects, the impatient Player 1 would trade all at once, or play his myopically best action. Instead he conceals his actions behind the background noise, but less willingly so at greater interest rates, when the future matters less.[3] The information burn rate is greatest when the market is most deceived — it explodes as the square root of the market deception, just like the value of disinformation. In the bounded case, we find that the monopolist maximally burns his information when the market deception passes a threshold.

Information is exploited more slowly when actions are more clearly observed, or the decision maker is more patient. Again, the square root law of comparative statics appears. When the interest rate scales by four, information is burned twice as fast and the value is divided in half — and not by four, as a bond would. In the limit of perfect impatience or observability, a dynamic "No Trade Theorem" arises. This famous result is seen in our repeated games setting as the famous 1960s information concealment result (Aumann, Machler, and Stearns 1995).

A key concept here is *liquidity*, roughly analogous to price elasticity. In our repeated games setting, this measures how elastic are beliefs in the informed player's strategy. An equilibrium notion, it is proportional to the square root of the observation process variance divided by the interest rate. Information is burned faster — namely, actions are less revealing — in a more liquid game. We find that liquidity varies inversely with the *ex ante* value of information, exploding near point beliefs.

[3] See Back and Baruch (2004) and Huddart, Hughes, and Levine (2001).

3. Market Rnesponses. Critical here is the nature of the information monopoly. The market price is an unbiased probability of the high state with unbounded burn rates. But with short-trade constraints, prices must systematically deviate from probabilities: Prices inflate when the market is optimistic and deflate when pessimistic. In the repeated game, the mixed strategy responses are analogously biased. The insight into the price deviation follows from mixed strategy logic: each side must mix to ensure the other is indifferent. When trading constraints bind, the market rationally expects the insider to trade less intensely. In turn, this makes the market less sensitive to observed trades, and this increased liquidity raises the payoff to insider trading. As the trading constraint only binds in one state, the market price must adjust to ensure indifference across all trading rates for the insider in the non-binding state. For example, when the market grows quite convinced that the state is low, the trading constraint does not bind in the low state when the insider is a seller. So the market price adjusts downward to reduce the payoff and ensure indifference in the low state.

The above logic implies that the market rationally expects extreme prices to mean revert. This occurs because the Bayesian market probability is a martingale, and high prices are a concave function of the probability and low prices are a convex function of the probability. We show that this finance notion applies in game theory: mixed strategies mean revert in competitive games.

4. Monopoly Durations. While information monopolies are ephemeral, we find that they never fully expire. With unbounded burn rates, "time is money" — the passage of time passes roughly corresponds to the accumulation of wealth. The expected time until earning any fraction of the value is an increasing and concave function of the fraction: The first hour is worth slightly more than the second hour, for instance.

Despite the fact that time is money, a longer monopoly need not be a stronger one. Noisier observations increase values, but have no effect on the expected time to produce any fraction of the value. If the market is more deceived, then the monopoly is more profitable, but arrives faster at any given fraction of its wealth. This corresponds to the story of Rothschild, whose massive wealth accumulation occurred in approximately 24 hours. Only in the case when the monopolist gets more patient do strength and length of a monopoly coincide: He values later profits more heavily, and so optimally burns his information more slowly.

Assuming a Wiener process for noise traders is well-established. But modeling repeated games using Wiener noise was more recently done in a seminal paper Sannikov (2007). Its application in a reputation model of Faingold and Sannikov (2007) is technically closer in spirit to this model, given the adverse selection problem.

Our model with unbounded intensity traverses the road first traveled by Kyle (1985). His paper differs by assuming a fixed finite horizon after which the market closes. While he too has a Gaussian noise process, his state space is the continuum; this forces him to restrict to linear equilibria. Our paper determines the maximum rent extraction among (highly nonlinear) symmetric Markovian equilibria with two states, before showing how net order flows affects this answer.

There is a large literature on mean reversion in asset prices.[4] We believe that our explanation based on informational monopoly with limited positions is new.

We believe that our link between constrained trading games and standard games of incomplete information is new. This connection opens the door to exploration of known price patterns in terms of mixed strategies of repeated games.

The static value of information has long been studied in statistics and economics. Our strategic value of information has a slight ring of issues explored in Hirshleifer (1971) and Schlee (2001). They show how the value of information is quite different in an equilibrium context, since its release inhibits information sharing possibilities that it makes everyone worse off. Their framework did not involve market power like ours does, nor is it dynamic.

But to keep the focus on economics, we follow the trend of this literature and not weigh down the paper with technical assumptions. We do not think that this introduces any ambiguity or essential imprecision. We briefly summarize these in an appendix.

[4]Among some of the most well-cited papers, mean reversion arises due to an underlying mean-reverting process in Cecchetti, Lam, and Mark (1990) and Wang (1993), while de Long, Shleifer, Summers, and Waldmann (1990) show that price-chasing by feedback traders also gives it.

2 A Model of Informed Trade

A continuous time two player game is played on $[0, \infty)$. We begin in a trading context with two states of the world and an infinite time horizon. Player 1 is an insider or long-run player, and is apprised of the state. The asset is worth 0 and 1, respectively, in states $\theta = 0$ and $\theta = 1$ (prior $q(0)$). He chooses a *net order flow* volume $N(t)$ at each moment in time.[5] So the absolute net order flow $|N(t)|$ is the net purchase rate if $N(t) > 0$, and is the net sales rate if $N(t) < 0$. In the *unbounded intensity model*, the net purchase rate may be any real $N(t) \in \mathbb{R}$. In the *bounded intensity model*, the insider often can only take the limited position $|N(t)| \leq 1$. This is like the unit trade limit in the bid-ask spread model of Glosten and Milgrom (1985).

Meanwhile, player 2 is really a *sequence* of uninformed traders, apprised of neither the state nor the payoffs of past trades — just like market makers. Capturing the moral hazard element in this game, Player 2 cannot precisely descry the actions of Player 1. She sees the net order flow of Player 1 obscured by a Wiener noise trader process $W(t)$:

$$dY = N\ dt + \sigma\ dW. \tag{1}$$

From this, she computes the updated posterior belief $q(t)$ (done in §4). Since there is a sequence of uninformed players 2, we also call these the *market beliefs.*

This is a zero sum trading game between risk neutral players. Simultaneous to Player 1's choice[6] of time-t net purchase rate $N(t)$, Player 2 sets the *price* $p(t) \in [0, 1]$ having observed all order flows up to date t. Player 1's time t flow payoffs are

$$N(\theta - p) \qquad \text{state } \theta = 0, 1 \tag{2}$$

and of course, this is the negative of Player 2s payoffs. We relate our game structure to the market microstructure models of finance in §**??**.

The longrun Player 1 discounts future payoffs at the interest rate $r > 0$. For the trading context, there might no explicit impatience, but instead a constant rate r of market closure, with payoffs deferred until a stochastic conclusion time of the game.

[5]To this point, our model is similar to Back and Baruch's (2004) two state version of Kyle (1985).

[6]The distinction between sequential and simultaneous choices in this continuous setting is not important, since the choices have a continuous sample path.

3 The Model Reborn as a Repeated Game

A. A Zero-Sum Game. We now rethink the bounded trading model as a repeated competitive game in normal form. As a conceptual bridge, we consider first a simple binary action zero-sum normal form trading game. Here in Figure 1, the long-run Player 1 knows the state of the world — that the asset has a common value of either \$0 or \$1 to the players.

asset is worth $\theta = 0$ (chance $1-q$)

	price $= 1$	*price* $= 0$
buy	-1	0
sell	1	0
	p	$1-p$

asset is worth $\theta = 1$ (chance q)

	price $= 1$	*price* $= 0$
buy	0	1
sell	0	-1
	p	$1-p$

Figure 1: **Trading as a Zero Sum Binary Game.** Player 1's payoffs are indicated from buying or selling a single share at either \$0 (left) or \$1 (right).

The Nash equilibrium mixture weight on the action "price = \$1" is what we have called the price $p \in [0,1]$. The net order flow N is the chance of buying a share minus the chance of selling a share, and so is bounded above by 1. As in the trading game of §2, Player 1's payoffs are $-Np$ at left, and $N(1-p)$ at right. The *static* "value" of the game (the unique Nash payoff) is zero for both players. Player 2 is willing to randomize over prices when Player 1 chooses trading intensities so that

$$(1-q)N_0 + qN_1 = 0 \tag{3}$$

The repeated version of this game is much richer, and Player 1 earns information rents.

In the repeated game, Player 2 is still a sequence of short-run one-period players in the same strategic role. Each makes imperfect observations Y of Player 1's actions, obscured by a Gaussian noise process in (1). In so doing, *we reinterpret the earlier noise traders as observation errors.* Each also does not see payoffs of previous players 2 — the standard informational assumption of the herding literature, for instance.

B. Beyond Zero-Sum Games. Our analysis pushes outside the realm of constant sum games. If we fix Player 1's payoffs, then Player 2 can entertain any payoffs provided her indifference condition remains the same. This class of games is captured by the

$\theta = 0$

$(-1, 1+b)$	$(0, b)$
$(1, a-1)$	$(0, a)$

$\theta = 1$

$(0, A)$	$(1, A-1)$
$(0, B)$	$(-1, 1+B)$

Figure 2: **More General Stage Games.** Player 1 alone knows the game matrix. Player 1 chooses rows, and Player 2 (in sequence) chooses columns. Payoffs are given by the right (left) matrix in state $\theta = 1$ with chance q (state $\theta = 0$ with chance $1 - q$).

competitive game form of Figure 2. In this game, Player 1's payoffs are unchanged; meanwhile, Player 1 is informed of the game played, $\theta = 0, 1$. Capturing the adverse selection, he earns a bonus of 1 for coordinating his action with the state, by choosing actions down/up in states $\theta = 0, 1$, respectively. Reflecting the competitive nature of the game, Player 1 earns a bonus of 1 for mis-coordinating his action with Player 2, whereas Player 2 earns 1 util more for coordinating her action with Player 1. The extra flexibility allows that Player 2 might independently prefer one of the actions of Player 1 (so that $a \neq b$ or $A \neq B$), or one of the states (in which case, $a \neq A$ or $b \neq B$).

Assume that the long-run player is pitted against a sequence of opponents who may choose two different actions, and only imperfectly observe predecessors' actions. Opposing actions are optimal states 0 and 1. For instance, consider penalty kicks in a game of soccer (Chiappori, Levitt, and Groseclose 2002). Assume the kicker has some private information — a recent hidden foot injury may make it harder to kick to left. The kicker can choose left or right, but prefers right given the injury. The goalie only imperfectly observes the kicker's past history. He wishes to lean the direction that the kicker will aim for, while the kicker prefers the opposite. Then risk neutrality over the kick outcome corresponds to $A = a = B = b = 0$. We allow that depending on the score in the game, the goalie values the loss from a goal differently than the kicker does.

Next, consider instead this as a model of two long-run players. Here, let us assume that the payoffs are unobserved simply because they are deferred until the end. In the motivational war example of the D-Day invasion choice, the additional flexibility admits that when Normandy is the better invasion location, its advantages over Calais may outweigh Calais' over Normandy when it is the better locale.

C. Information Concealment and the Temptation to Chisel. For some insight into our dynamic game to come next, consider the induced one-shot game when the informed player does not exploit his information. The value of this game is 0 for

	price $= 1$	price $= 0$
B	$(q-1, qX+(1-q)y)$	$(q, qX+(1-q)y-1)$
S	$(1-q, qY+(1-q)x-1)$	$(-q, qY+(1-q)x)$

Figure 3: **Expected Payoffs with Symmetric Uncertainty.** The trading rates that leave Player 2 indifferent across prices is the same here as in the game of Figure 1.

all beliefs q. Consider the repeated game with observed actions, and no discounting. Since this is weakly a concave function of q, Aumann, Machler, and Stearns (1995) prove that the best commitment mixed strategy is not to reveal any information: If Player 1's equilibrium trading rate differs in the states, then the true state is almost learned in finite time. Since this either hurts one type of Player 1, or is payoff-neutral, this strategy need not be used in equilibrium.

By contrast, our context assumes partially-observed actions, and payoff discounting. The impatient Player 1 is now tempted to play his dominant strategy, while the noise means that he need only slightly chisel away his reputation. He then plays his preferred action slightly more often. For if the market expects the same actions in both states, then intuitively it does not update beliefs given any observations. In fact, this logic holds at any market belief, and thus eventually the information advantage should vanish. (Both claims are rigorous, as we see in (5).) These are key findings of the reputation literature,[7] where one type of Player 1 has a dominant strategy. Here, neither type of Player 1 has a dominant strategy. In the next section, we precisely characterize this information exploitation, and then flesh out its implications.

4 Market Beliefs

We restrict focus to *Markov equilibria*, in which both players' strategies depend on the current market belief q_t (Player 2's posterior) in state $\theta = 1$. Arising from public observations, this belief is common knowledge, and so a natural state variable for us.

Player 2 expects the net order flow n_θ in state θ. Her expectation is correct in equilibrium, namely $N = n_\theta$ in state-θ. Since Player 2 cannot affect the belief evolution,

[7]See Cripps, Mailath, and Samuelson (2004), a key sequel to Fudenberg and Levine (1992).

she acts myopically. In light of (3), she expects the net order flow drift

$$n(q) \equiv (1-q)n_0(q) + qn_1(q) = 0 \tag{4}$$

Suppressing the q argument, Appendix A.1 derives the process of market beliefs:

$$dq = \left[q(1-q)(n_1-n_0)(N-n)/\sigma^2\right] dt + \left[q(1-q)(n_1-n_0)/\sigma\right] dZ. \tag{5}$$

where Z is a Wiener process.[8] This belief process obtains in and out of equilibrium, and has many intuitive properties. First, volatility rises with more precise action observations (smaller σ^2). Or, if Player 2 expects the same net order flow in the two states, then she rationally ignores the innovations in the signal Y and her beliefs cannot change. This also holds if Player 1 chooses the net order flow $N = n$. Also, as a function of the realized net order flow N, the belief drift is linear and its volatility constant.

5 Equilibrium Analysis

A. Unbounded Trades. We first solve the unbounded intensity trading model, and then build on its solution for the general case. Only Player 1 engages in a far-sighted optimization. Let $V_\theta(q)$ be the present value of payoffs to Player 1 in state θ given the belief q. The two Bellman equations as usual can be intuitively encapsulated by the truism that "the return equals the dividend plus the expected capital gain". In our case, the dividend is the flow payoff (2), and capital gains are governed by the drift and variance of the process (5). Multiplication by σ^2 conveniently yields the recursion:

$$r\sigma^2 V_\theta \;=\; \max_N N(\theta-p)\sigma^2 + q(1-q)(n_1-n_0)(N-n)V_\theta' + \frac{1}{2}q^2(1-q)^2(n_1-n_0)^2 V_\theta''$$

So Player 1 chooses a best response to the price. Absent trading constraints, this is a mixed strategy Nash equilibrium at any moment in time. Player 1 must be indifferent across all net order flows in equilibrium, as his payoff is linear in his strategy. Let's write the price $p(q)$ as a function of the probability q of state $\theta = 1$. Then the first

[8] This process is formed from the market's perspective: It is an expectation of the two possible Wiener processes W, each consistent with one of the two states $\theta = 0, 1$ in equilibrium.

order condition only reflects impatience through its effect on the state θ value function:

$$0 = \theta - p(q) + q(1-q)(n_1(q) - n_0(q))V_\theta'(q)/\sigma^2 \tag{6}$$

Substituting this first order condition into the Bellman equation yields:

$$r\sigma^2 V_\theta(q) = -n(q)q(1-q)(n_1(q) - n_0(q))V_\theta'(q) + \frac{1}{2}q^2(1-q)^2(n_1(q) - n_0(q))^2 V_\theta''(q) \tag{7}$$

We see here how impatience and observation noise always interact multiplicatively. Increasing either allows the informed player to exploit his advantage more prudently.

At centerstage in our equilibrium description is the *Probit function* $F(q)$, i.e., the inverse of the standard normal distribution function. We now derive the *strategic value of disinformation*, or the loss of being the uninformed party, given the state.

Theorem 1 (*a*) *There is a unique Markov equilibrium with unbounded trades.* (*b*) *The equilibrium strategies entail the price* $p(q) = q$ *and the trading intensities*

$$\frac{\sigma\sqrt{2r}}{qF'(q)} = n_1(q) > 0 > n_0(q) = -\frac{\sigma\sqrt{2r}}{(1-q)F'(q)} \tag{8}$$

(*c*) *The values in the two states are related by* $V_0(1-q) = V_1(q)$, *where*

$$V_0(q) = \frac{\sigma}{\sqrt{2r}}\left[qF(q) + \frac{1}{F'(q)}\right]$$

Our existence argument is constructive. We deduce uniqueness in value space using the zero sum structure; strategies follow by differentiation. Since Player 2 acts myopically, it may seem intuitive that she sets the price $p(q) = q$; however, her myopia only affects Player 1's trading intensity (4). Notice however that with this price, the longrun Player 1 would earn zero profits were he uninformed: For with chance q, buying would gain $1 - p(q)$ per share, and with chance $1 - q$, it would lose $p(q)$ per share. Thus, an uninformed Player 1 would be indifferent about buying or selling any number of shares. This ensures that all profits are a pure rent to the insider information.

B. Bounded Trades. With a maximal unit net order flow, or in the equivalent repeated game model, the earlier Bellman equation must include the constraint $|N| \leq 1$. We find that the trading constraint binds on Player 1 whenever Player 2 is sufficiently deceived. If Player 1 is patient enough, or his actions are well-enough observed, there exists an interior *confounding interval* inside which Player 1's absolute net order flow is less than one in both states. Specifically, this confounding region is nonempty when the precision $1/\sigma^2$ of Player 2's action observations exceeds the discount rate r. In the repeated game context, his mixed strategy is interior. Outside this interior region, the price leans toward the more likely state.

Theorem 2 (*a*) *There is a unique Markov equilibrium with bounded trades.* (*b*) *The net order flow is lower than in the unbounded model at all beliefs q.* (*c*) *If $r\sigma^2 \geq 1$ then $n_1(q) = 1$ for $q \geq 1/2$ and $n_0(q) = -1$ for $q \leq 1/2$.* (*d*) *If $r\sigma^2 < 1$, then $\exists$ an increasing function $q^*(r\sigma^2) \in (0, 1/2)$, so that*

- *For all beliefs $q < q^*$, we have $n_0(q) = -q/(1-q)$ and $n_1(q) = 1$*
- *For all beliefs $q^* < q < 1 - q^*$, we have $-1 < n_0(q) < 0 < n_1(q) < 1$*
- *For all beliefs $q > 1 - q^*$, we have $n_0(q) = -1$ and $n_1(q) = (1-q)/q$*

(*e*) *The price is $p(q) = q$ for $q \in (q^*, 1-q^*)$, it is $p(q) < q$ for $q < q^*$, and finally it is $p(q) > q$ for $q > 1 - q^*$. When $r\sigma^2 > 1$ the price jumps up discretely at $q = 1/2$.*

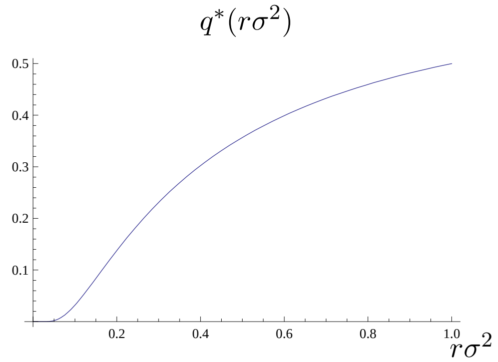

Figure 4: **Comparative Statics of q^* in $r\sigma^2$.**

6 "Square Root Law" of Comparative Statics

> *"The art of using deceit and cunning grow continually weaker and less effective to the user."* — John Tillotson, Archbishop of Canterbury (1691–94)

We now flesh out key properties of the net order flow. How should trading intensities optimally adjust in response to greater impatience, noise, or deception? Or in the repeated normal form game, how defensive should the informed player act? Barring effects on the price, the impatient Player 1 would trade all at once. Instead he conceals his trades behind the background noise. As the interest rate rises, the cost of delaying trade rises, and trades behave (8) like $\sqrt{r}$. This "substitution effect" mitigates impatience losses; on balance, the disinformation values fall like $1/\sqrt{r}$, as opposed to the bond price $1/r$.

This has an option value interpretation: Greater r leads to smaller V_θ'' so that the option value to delay falls as r rises. The first effect dominates, but the second reduces the burn rates response to r. By the same token, the cost of trading falls in the noise level σ^2, and trades (8) fall in proportion to σ. On balance, the disinformation values fall in proportion to $1/\sigma$.

The above belief formulas in terms of the probit function admit more transparent expressions near extremes near 0 or 1. In state $\theta = 0$, as the market grows more deceived,

$$n_0(q) \propto \sqrt{\ell(q)} \qquad \text{and} \qquad V_0(q) \propto \sqrt{\ell(q)} \qquad \text{as } q \to 1$$

So a marketplace that is four times as deceived (in terms of ℓ) will see this deception only exploited twice as intensely and the resulting profits will be twice as high. Since there are diminishing returns to greater deception, but constant marginal costs of deception, activities like those attributed to Rothschild.

Asymptotic behavior of strategies:

$$n_1(q) \sim 2q\sigma\sqrt{2\pi r \ell(q)} \quad q \to 0 \qquad n_0(q) \sim -2\sigma\sqrt{-2\pi r \ell(q)} \quad q \to 1.$$

7 Mean Reversion of Prices and Strategies.

We turn to our most striking finding with bounded trades. The confounding set exists when Player 1 is patient enough or noise trade is small (or actions well-observed in a repeated game). On this confounding set, prices equal expected values. Outside this

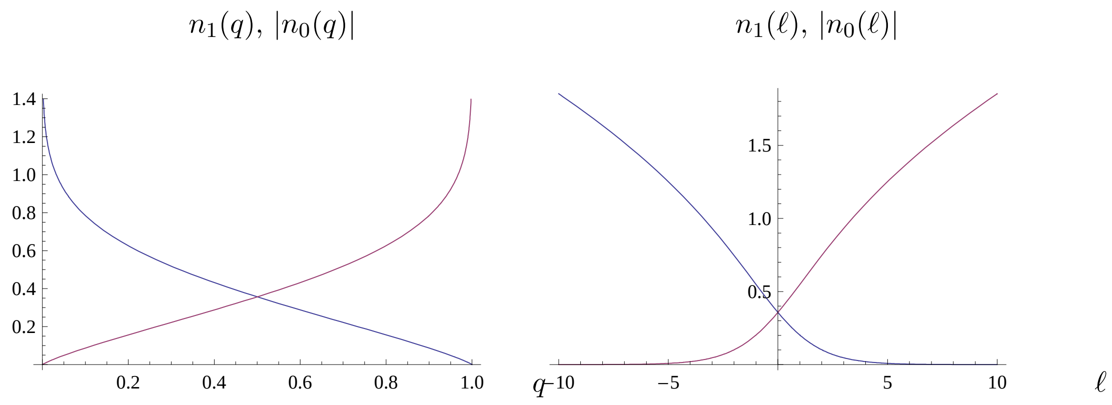

Figure 5: **Equilibrium Net Order Flows:** n_1 (blue) and $|n_0|$ (red), given $r = 0.1$ and $\sigma = 1$. The graph on the left is in belief space q, while the graph on the right is in log-likelihood space ℓ.

interval, the price favors the more likely state, generating rational asset bubbles. The intuition for price deviating from expected value is as follows:

- Let $q < q^*$ so that the constraint binds in state $\theta = 1$:
  - With an upper bound on n_1, the market expects lower $(n_1 - n_0)$.
  - Thus beliefs are less sensitive to trades (liquidity rises).
  - With higher liquidity, sales in state $\theta = 0$ have a smaller impact on q.
  - To maintain the informed player's indifference in state $\theta = 0$, the flow benefit to selling must fall, which requires a lower price.
- Symmetric reasoning holds for $q > q^*$, save $\uparrow p$ to lower the flow benefit to selling.

Not only can we show that prices are biased toward the more likely state: prices are a convex function of beliefs below q^* and a concave function of beliefs above q^*. Mean reversion then follows from Ito's Lemma: $E[dp] = p'(q)E[dq]+p''(q)E[dq^2]/2$ and beliefs being a martingale ($E[dq] = 0$). Likewise, mixed strategies of the uninformed Player 2 mean revert in competitive games. *This is a novel explanation for mean reversion in asset prices that extends to mean reversion in market strategies.*

Moreover, we can show that the *price deviation is aggravated by greater noise σ^2 or impatience r.* By Theorem 2, the region in which the price deviation occurs increases.

Moreover, absolute expected price change and price deviation $|p(q) - q|$ increases in this region when σ^2 or r increases. When $r\sigma^2 > 1$ then the interior confounding set vanishes, trading constraints always bind, and the price jumps up discretely at $q = 1/2$.

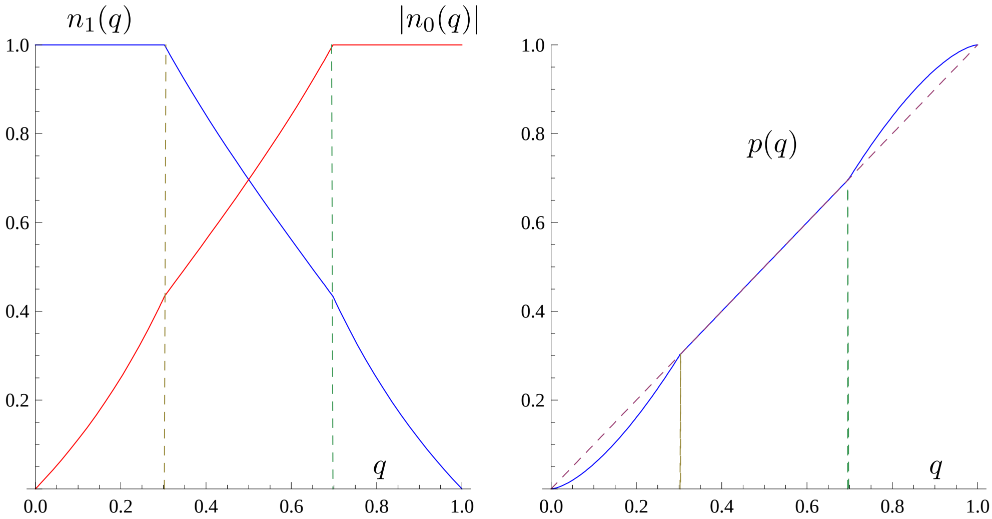

Figure 6: **Strategies with Flow Capacity Constraints:** The left graph depicts the absolute value of the net order flows $N_1(q)$ and $|n_0(q)|$. The graph on the right compares the uninformed strategy $p(q)$ to the 45 degree line (for $r = 0.1$ and $\sigma = 2$).

8 Time and Money

- Substituting unconstrained equilibrium strategies into the market belief evolution yields equilibrium drift $\mu_\theta(q)$ and variance $\varsigma^2(q)$:

$$\mu_1(q) = \frac{2r}{qF'(q)^2} = -\mu_0(1-q) \quad \text{and} \quad \varsigma^2(q) = \frac{2r}{F'(q)^2}.$$

- When trading rates are unconstrained, the equilibrium belief process is proportional to the interest rate, but does not depend on noise. As noise increases the informed player increases his trading rate exactly to offset the decreased observability of trades.

- In contrast, in the trade constrained region the drift and variance of beliefs is proportional to σ^{-2}, but not a function of the interest rate.

- The truth is revealed in the limit, but not in any finite time. While information monopolies are ephemeral, our boundary analysis of the equilibrium belief diffusion reveals that they never fully expire.

- Let $T(q)$ be the expected time until the belief process leaves any strict subset of $[0, 1]$. By Ito's Lemma this obeys:

$$-1 = \frac{2r}{F'(q)^2} T''(q) \tag{9}$$

- Information rent is symmetric about $q = 1/2$. So is the expected time to leaving any interval $(a, 1-a)$. Given this co-monotonicity, define $T(V(q))$. We show in Appendix that for the unbounded trades case:

$$rT(\nu) = \log(\nu) \sum_{n=0}^{\infty} \frac{a_n \left[\log(\nu)\right]^n}{n!} + \text{constant}$$

where $\nu = V/V(1/2)$. We graph this function in Figure 7.

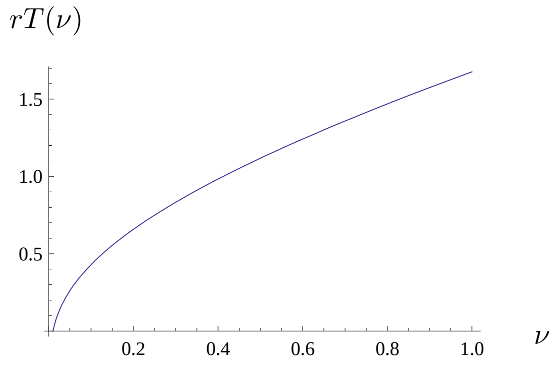

Figure 7: Time as a function of money (percentiles).

- The expected time until earning any fraction of the value is an increasing and concave function of the fraction: The informed burns his information rent faster the higher it is. When ν is not too small this function is approximately linear, and so "time is money."

- Time is money, but a longer monopoly need not be a stronger one. Noisier observations increase values, but have no effect on the expected time to produce any fraction of the value. If the market is more deceived, then the monopoly is more profitable, but arrives faster at any given fraction of its wealth. Only when the monopolist gets more patient do strength and length of a monopoly coincide.

A Analysis Details

A.1 Belief Processes

Consider the change in beliefs from $Q(t) = q$ over $[t, t + \Delta t]$. Let ΔY be the change in the signal process Y over the interval $[t, t + \Delta t]$ observed by the uninformed player. Then by Bayes' rule:

$$\Delta Q \equiv Q(t + \Delta t) - q = \frac{q(1-q)(f_1(\Delta Y) - f_0(\Delta Y))}{q f_1(\Delta Y) + (1-q) f_0(\Delta Y)}$$

where $f_\theta(\Delta Y)$ is the uninformed player's belief about the density over ΔY. From the point of view of the uninformed player, Y is a Brownian motion with variance σ^2 and state conditional drift n_θ. Thus, as is well known, ΔY will be normally distributed with mean $n(q)dt$ and variance $\sigma^2 \Delta t$, so that:

$$f_\theta(\Delta Y) \equiv \frac{1}{\sigma\sqrt{2\pi\Delta t}} e^{-\frac{1}{2\sigma^2\Delta t}(\Delta Y - n(q)\Delta t)^2}.$$

Factoring out common terms in $f_\theta(\Delta Y)$ and simplifying yields:

$$\Delta Q = \frac{q(1-q)\left[g_1(\Delta Y) - g_0(\Delta Y)\right]}{q g_1(\Delta Y) + (1-q) g_0(\Delta Y)}$$

where

$$g_\theta(\Delta Y) \equiv e^{(n_\theta(q)\Delta Y - n_\theta(q)^2 \Delta t/2)/\sigma^2}$$

Substituting the second order Taylor series approximation $e^z = 1 + z + z^2/2$ near $z = 0$ yields:

$$g_\theta(\Delta Y) \approx 1 + (n_\theta(q)\Delta Y - \tfrac{1}{2} n_\theta(q)^2 \Delta t)/\sigma^2 + \tfrac{1}{2}(n_\theta(q)\Delta Y - \tfrac{1}{2} n_\theta(q)^2 \Delta t)^2/\sigma^4$$

While the density $f_\theta(\Delta Y)$ depends on the beliefs of the uninformed player, the realized ΔY will depend on the actual drift N: $\Delta Y = N\Delta t + \sigma \Delta W$. Making this substitution and canceling all terms above the first order in Δt (recalling ΔW is of the order $\sqrt{\Delta t}$), we find:

$$g_\theta(N\Delta t + \sigma\Delta W) \approx 1 + n_\theta(q)\left[N\Delta t + \sigma\Delta W\right]/\sigma^2$$

Now substitute back to the ΔQ equation to get:

$$\begin{aligned}
\Delta Q &\approx \frac{q(1-q)(n_1(q)-n_0(q))\left[N\Delta t + \sigma\Delta W\right]/\sigma^2}{1+n(q)\left[N\Delta t + \sigma\Delta W\right]/\sigma^2} \\
&\approx q(1-q)(n_1(q) - Nn_0(q))\left[N\Delta t + \sigma\Delta W\right]/\sigma^2\left[1 - n(q)\left[N\Delta t + \sigma\Delta W\right]/\sigma^2\right] \\
&\approx q(1-q)(n_1(q)-n_0(q))\left[(N-n(q))\Delta t + \sigma\Delta W\right]/\sigma^2
\end{aligned}$$

where we have again used $(\Delta W)^2 \approx \Delta t$ and canceled all terms above the first order in Δt. Then:

$$E[\Delta Q] = q(1-q)(n_1(q)-n_0(q))(N-n(q))/\sigma^2\Delta t$$

and

$$Var[\Delta Q] = q^2(1-q)^2(n_1(q)-n_0(q))^2/\sigma^2\Delta t$$

A.2 Equilibrium Construction: Unconstrained Solution

We shall verify that our proposed solution is an equilibrium whenever trading constraints do not bind, and so solves the Intensity model.

Step 1: A Differential Equation for Values and strategies.

Define $\Delta(q) \equiv V_0(q) - V_1(q)$. Equate the price from the first order conditions (6):

$$n_1(q) - n_0(q) = \frac{\sigma^2}{q(1-q)\Delta'(q)} \tag{10}$$

Substituting this and Player 2 indifference (4) into the Bellman equation (7) yields:

$$(r/\sigma^2)V_\theta(q) = \frac{V_\theta''(q)}{2\Delta'(q)^2} \tag{11}$$

Differencing this at $\theta = 0, 1$ yields a second order ordinary differential equation for Δ:

$$2(r/\sigma^2)\Delta(q) = \frac{\Delta''(q)}{\Delta'(q)^2} \tag{12}$$

Substitute (10) into Player 2's indifference equation (4) and the FOCs (6), to get:

$$n_0(q) = -\frac{\sigma^2}{(1-q)\Delta'(q)} \quad \text{and} \quad n_1(q) = \frac{\sigma^2}{q\Delta'(q)} \quad \text{and} \quad p(q) = \theta + \frac{V_\theta'(q)}{\Delta'(q)} \tag{13}$$

Altogether we may finish the construction by verifying in turn: the modified State 0 Bellman equation (11); the modified FOCs in (13); and our proposed solution to (12).

Henceforth, let us replace $r \leftarrow r/\sigma^2$.

Step 2: If (12) holds, then $V_0(q) = q\Delta(q) + [2r\Delta'(q)]^{-1}$ satisfies (11).

Differentiating the suggested V_0 function to get:

$$V_0'(q) = \Delta(q) + q\Delta'(q) - \frac{\Delta''(q)}{2r\Delta'(q)^2}.$$

Combining this with equation (12) and simplify to discover:

$$V_0'(q) = q\Delta'(q). \tag{14}$$

Differentiating (14), dividing both sides by $2r\Delta'(q)^2$, and using (12), yields the desired:

$$\frac{V_0''(q)}{2r\Delta'(q)^2} = \frac{q\Delta''(q)}{2r\Delta'(q)^2} + \frac{1}{2r\Delta'(q)}.$$

Step 3: If $V_1(1-q) = V_0(q) = q\Delta(q) + [2r\Delta'(q)]^{-1}$, then $p(q) = q$ uniquely satisfies both FOCs in (13).

That $p(q) = q$ uniquely satisfies the 0 State FOC in (13) trivially follows from (14). By definition of $\Delta(q)$ and Step 2, we have:

$$V_1(q) = -(1-q)\Delta(q) + \frac{1}{2r\Delta'(q)}$$

Differentiating this solution and using equation (12), yields $V_1'(q) = -(1-q)\Delta'(q)$, which implies that $p(q) = q$ solves the 1 State FOC in (13).

Step 4: If $\Delta(1/2) = 0$, then (12) has the general solution

$$\Delta(q) = (2r)^{-1/2} F\left(\tfrac{1}{2} + \tfrac{1}{2}c\left(q - \tfrac{1}{2}\right)\right)$$

where $F(q)$ is the inverse of the standard normal CDF. If the informed player's value is unbounded as the uninformed player becomes sufficiently convinced in a lie (eg., $\lim_{q \to 1} V_0(q) = \infty$), then $\Delta(q) = (2r)^{-1/2}F(q)$.

Substitute $x = q - 1/2$ and $H(x) = \Delta(x + 1/2)$ into (12) to get:

$$2rH(x) = \frac{H''(x)}{H'(x)^2}.$$

One can verify that the general solution to this homogenous differential equation is:

$$H(x) = r^{-1/2}\mathrm{erf}^{-1}(c_1 x + c_1 c_2),$$

where $\mathrm{erf}^{-1}(z)$ is the inverse of the error function $\mathrm{erf}(s) = 2[\int_0^s e^{-t^2} dt]/\pi$. Imposing $\Delta(1/2) = 0$ (i.e. $H(0) = 0$) implies $c_2 = 0$, and we get:

$$H(x) = r^{-1/2}\mathrm{erf}^{-1}(c_1 x).$$

Finally, substitute back in $x = q - 1/2$ to get: $\Delta(q) = r^{-\frac{1}{2}}\mathrm{erf}^{-1}(c(q - 1/2))$. To complete the proof use the identity $F(z) = \sqrt{2}\,\mathrm{erf}^{-1}(2z - 1)$.

The boundary condition $\lim_{q\to 1} \Delta(q) = \lim_{x \to 1/2} H(x) = \infty$ further restricts the parameter c. Use the identity $\mathrm{erf}^{-1}(\mathrm{erf}(s)) = s$ to translate this condition to:

$$\mathrm{erf}^{-1}(\mathrm{erf}(s)) = \infty \Rightarrow s = \infty.$$

Since $\mathrm{erf}(\infty) = 1$, this implies that $\lim_{x\to 1/2} cx = 1$ or $c = 2$. $\square$

Altogether in the unconstrained model we have: [9]

Values:	$V_0(q) = \frac{1}{\sqrt{2r}}\left[qF(q) + \frac{1}{F'(q)}\right]$	$V_1(q) = V_0(1-q)$	
Strategies:	$n_0(q) = -\frac{\sigma\sqrt{2r}}{(1-q)F'(q)}$	$n_1(q) = \frac{\sigma\sqrt{2r}}{qF'(q)}$	
Beliefs:	$\mu_1(q) = \frac{2r}{qF'(q)^2}$,	$\mu_0(q) = -\frac{2r}{(1-q)F'(q)^2}$	$\varsigma^2(q) = \frac{2r}{F'(q)^2}$

A.3 Equilibrium Construction: Bounded Model

We establish the result for $q \leq 1/2$, the analysis is symmetric for $q \geq 1/2$.

STEP 1: INFORMATION BURN REGION: VALUES, FOCS, AND PRICING

Substituting $n_1(q) - n_0(q) = (1-q)^{-1}$ into the $\theta = 0$ state FOC yields:

$$p(q) = qV_0'(q)/\sigma^2. \tag{15}$$

For $n_1(q) = 1$ to be optimal, the FOC in State 1 must be weakly positive. Substituting $n_1(q) - n_0(q) = (1-q)^{-1}$ into this FOC and substituting out the price using (15) yields:

$$FOC_1(q) \equiv 1 - \frac{q}{\sigma^2}\Delta'(q) \geq 0 \tag{16}$$

Substituting (15) plus $n_1(q) = 1$ and $n_0(q) = -q/(1-q)$ into the Bellman equations:

$$\begin{aligned} r\sigma^2 V_0(q) &= \tfrac{1}{2}q^2 V_0''(q) \\ r\sigma^2 V_1(q) &= \sigma^2 - qV_0'(q) + qV_1'(q) + \tfrac{1}{2}q^2 V_1''(q) \end{aligned}$$

This system has the following bounded solution:

$$V_0(q) = k_0 q^{\frac{1}{2}(1+A(r\sigma^2))} \quad \text{and} \quad V_1(q) = \tfrac{1}{r} + (k_1 + k_0 q)q^{\frac{1}{2}(A(r\sigma^2)-1)} \tag{17}$$

[9] Substituting the equilibrium strategies for N into the belief diffusion (5) produces the variance coefficient $\sigma/\Delta'(q)$ and the state contingent drifts:

$$\mu_0(q) = -\frac{\sigma^2}{(1-q)\Delta'(q)^2} \quad \text{and} \quad \mu_1(q) = \frac{\sigma^2}{q\Delta'(q)^2}$$

where k_0 and k_1 are undetermined constants and $A(r\sigma^2) = \sqrt{1 + 8r\sigma^2}$.

Substitute the $V_0(q)$ solution into equation (15) to get:

$$p(q) = \frac{k_0}{2\sigma^2} \left(1 + A(r\sigma^2)\right) q^{\frac{1}{2}(1+A(r\sigma^2))}. \tag{18}$$

By construction, as long as this equation is satisfied the FOC in State 0 holds. This will be feasible as long as the implied price is between 0 and 1. Direct calculation yields $p(0) = 0$. In our constructed equilibrium $k_0 > 0$ (as we show below), so $p'(q), p''(q) > 0$. Symmetric reasoning establishes $p(1) = 1$. Thus, $p(q)$ is continuous and increasing with endpoints $p(0) = 0$ and $p(1) = 1$. Altogether, as long as (18) obtains the State 0 FOC is satisfied and the implied price is feasible.

Altogether, on the interior confounding region, $p(q) = q$ and the value functions are determined up to the parameter c, as established in Appendix A.2, while the value functions and price in lower constrained region $q < q^*$ are determined up to the constants k_0 and k_1 as given by (17) and (18). In addition, we must check that the complementary slackness condition (16) holds on $q < q^*$.

The characterization of the equilibrium differs for $r\sigma^2$ greater or less than one. First we turn to the case in which no interior confounding equilibrium exists.

Step 2: If $r\sigma^2 > 1$ then $q^* = 1/2$,

$$k_0 = \frac{A(r\sigma^2) - 1}{r(1 + A(r\sigma^2))} 2^{\frac{1}{2}(A(r\sigma^2)-1)} \quad \text{and} \quad k_1 = -\tfrac{1}{r} 2^{\frac{1}{2}(A(r\sigma^2)-1)}.$$

In this equilibrium, the price $p(q)$ jumps discretely up at $q = 1/2$ and the gap $|p(q) - q|$ grows in $r\sigma^2$.

Step 2-a: Value Matching and Smooth Pasting are Satisfied.

Since $V_0(1/2) = V_1(1/2)$ and $V_0'(1/2) = -V_1'(1/2)$, when $q^* = 1/2$, value matching and smooth pasting in one state implies the value matching and smooth pasting in the other state. The given k_0 and k_1 uniquely solve smooth pasting and value matching when $q^* = 1/2$.

Step 2-b: Complementary Slackness is Satisfied.

Substituting these values, we find $FOC_1(1/2) = 1$. Further $k_1 < 0 \Rightarrow$

$$FOC_1'(q) = \frac{k_1 \left(A(r\sigma^2) - 1\right)^2}{4\sigma^2} q^{\frac{1}{2}(A(r\sigma^2)-3)} < 0,$$

and so $FOC_1(q) \geq 1$ for all $q < q^* = 1/2$.

Step 2-c: Price Characterization.

That $p(q) < q$ for $q \leq 1/2$ follows from (18) and the given k_0.

Now we establish that $|p(q) - q|$ increases in $r\sigma^2$ when $q < 1/2$. As we have shown $p(q) < q$ we do this by showing that $p(q)$ decreases in r:

$$\frac{\partial p(q)}{\partial r} = \frac{2^{\frac{1}{2}(A(r\sigma^2)-3)} q^{\frac{1}{2}(1+A(r\sigma^2))}}{r^2\sigma^2 A(r\sigma^2)} g(r\sigma^2, q)$$

where

$$g(r\sigma^2, q) = A(r\sigma^2) - 1 + r\sigma^2 \left((A(r\sigma^2) - 1)\log(4) - 4\right) + 2r\sigma^2 \left(A(r\sigma^2) - 1\right) \log q$$

and thus the sign of the derivative of $p(q)$ is the same as the sign of g, so we must show that $g(r\sigma^2, q) < 0$ for all $r\sigma^2 > 1$ and $q \leq 1/2$. This follows from

$$g_q(r\sigma^2, q) = 2r\sigma^2(A(r\sigma^2) - 1)/q > 0 \quad \text{and} \quad g(r\sigma^2, 1/2) = A(r\sigma^2) - 4r\sigma^2 - 1 < 0.$$

Step 3: If $r\sigma^2 < 1$, then we have:

$$\begin{aligned}
q^*(r\sigma^2) &= \frac{D(r\sigma^2)e^{-B(r\sigma^2)^2}}{2\left(D(r\sigma^2)e^{-B(r\sigma^2)^2} - \operatorname{erf}(B(r\sigma^2))\right)} \in (0, 1/2) \\
c^*(r\sigma^2) &= 2\left(D(r\sigma^2)e^{-B(r\sigma^2)^2} - \operatorname{erf}(B(r\sigma^2))\right) \in (2, \infty) \\
k_0 &= \frac{2\sigma^2}{1 + A(r\sigma^2)} (q^*)^{\frac{1}{2}(1-A(r\sigma^2))} > 0 \\
k_1 &= \frac{2\sigma^2}{A(r\sigma^2) - 1} (q^*)^{\frac{1}{2}(1-A(r\sigma^2))} < 0
\end{aligned}$$

where:

$$B(r\sigma^2) \equiv \frac{A(r\sigma^2) - 3}{4\sigma\sqrt{r}} \quad \text{and} \quad D(r\sigma^2) \equiv \frac{2\sigma\sqrt{r}}{\pi}.$$

Proof: Step 3-a: The Given (q^*, c^*) uniquely solve:

$$\lim_{q \downarrow q^*} V_0'(q) = \sigma^2 \quad \text{and} \quad \lim_{q \downarrow q^*} \Delta(q) = \frac{A(r\sigma^2) - 3}{4r}. \tag{19}$$

Substituting the probit function and it's derivative into system (19) and rearranging, this system becomes:

$$\text{erf}^{-1}(c(q - 1/2)) = B(r\sigma^2) \quad \text{and} \quad qc\ e^{\text{erf}^{-1}(c(q-1/2))^2} = D(r\sigma^2)$$

Note that $r\sigma^2 < 1 \Rightarrow B(r\sigma^2) < 0$.

Suppressing the exogenous arguments of the constants $B(\cdot)$ and $D(\cdot)$ and inverting the first equation we find:

$$c(q - 1/2) = \text{erf}(B) \Leftrightarrow q = q_1(c) \equiv 1/2 + \text{erf}(B)/c.$$

Using the first equation, substitute out for $\text{erf}^{-1}(c(q - 1/2))$ in the second equation to get:

$$qc\ e^{B^2} = D \Leftrightarrow \log(qc) + B^2 = \log(D) \Leftrightarrow q \equiv q_2(c) = \tfrac{1}{c} De^{-B^2}.$$

Now set $q_1(c) = q_2(c) \Leftrightarrow$

$$c^* = 2\left(De^{-B^2} - \text{erf}(B)\right) \in (2, \infty) \quad \Rightarrow \quad q^* = \frac{De^{-B^2}}{2\left(De^{-B^2} - \text{erf}(B)\right)} \in (0, 1/2).$$

where the bounds on q^* follow from $r\sigma^2 \in (0, 1) \Rightarrow B < 0 \Rightarrow \text{erf}(B) < 0$. This also implies that $c^* > 0$. To establish the lower bound on c^*, we need:

$$h(x) \equiv D(x)e^{-B(x)^2} - \text{erf}(B(x)) > 1 \quad \forall\ x \in (0, 1),$$

which follows from

$$h'(x) = e^{-B(x)^2}\left[\frac{(2x + 1)A(x) - 6x - 1)}{2x^{3/2}A(x)\sqrt{\pi}}\right] > 0 \quad \text{and} \quad h(0) = 1.$$

Step 3-b: The CS Condition (16) obtains and $FOC_1(q^*) = 0$.

Straightforward algebra establishes that $\lim_{q \uparrow q^*} FOC_1(q) = 0$ evaluated at the given

k_1. The CS condition holds for all $q < q^*$, since $r\sigma^2 < 1 \Rightarrow k_1 < 0 \Rightarrow$

$$FOC_1'(q) = \left(\frac{1 + 4r\sigma^2 - A(r\sigma^2)}{2\sigma^2}\right) k_1 q^{\frac{1}{2}(A(r\sigma^2)-3)} < 0.$$

STEP 3-C: THE SMOOTH PASTING CONDITIONS ARE SATISFIED.

Routine algebra yields $\lim_{q\uparrow q^*} V_0'(q^*) = \sigma^2$ for the given k_0, which establishes smooth pasting in State 0 by Step 3-A. Again routine algebra establishes $q^* \lim_{q\uparrow q^*} \Delta(q) = \sigma^2$, so together we have both:

$$\lim_{q\uparrow q^*} V_0'(q) = \lim_{q\downarrow q^*} V_0'(q) \quad \text{and} \quad \lim_{q\uparrow q^*} V_0'(q) = q^* \lim_{q\uparrow q^*} \Delta'(q).$$

In addition, by Equation (14) we have $\lim_{q\downarrow q^*} V_0'(q) = q^* \lim_{q\downarrow q^*} \Delta'(q)$. Altogether these three conditions establish that $\lim_{q\uparrow q^*} \Delta'(q) = \lim_{q\downarrow q^*} \Delta'(q)$.

STEP 3-D: THE VALUE MATCHING CONDITIONS ARE SATISFIED.

Substituting the given k_0 and k_1 we have:

$$\lim_{q\uparrow q^*} \Delta(q) = \frac{A(r\sigma^2) - 3}{4r} = \lim_{q\downarrow q^*} \Delta(q) \quad \text{(by Step 3-A)}.$$

To establish value matching for the $\theta = 0$ state, substitute $\lim_{q\downarrow q^*} \Delta(q) = \frac{A(r\sigma^2)-3}{4r}$ (by Step 3-A) and $\lim_{q\downarrow q^*} \Delta'(q) = \sigma^2/q^*$ (by Step 3-A and (14)) into the confounding region value function $V_0(q) = q\Delta(q) + [2r\Delta'(q)]^{-1}$ to get:

$$\lim_{q\downarrow q^*} V_0(q) = \frac{2q^*\sigma^2}{1 + A(r\sigma^2)}$$

Then substitute the given k_0 into the $V_0(q)$ equation in (17), which yields the same function of q^*. □

A.4 Convergence Rates

UNCONSTRAINED MODEL: There are a few references for the asymptotic behavior of $F(q)$ as $q \to 0$ or $q \to \infty$, we use Dominici (2003). He reports:

$$\begin{aligned} F(q) &\sim -\sqrt{-\log\left(2\pi q^2\right) - \log\left(-\log\left(2\pi q^2\right)\right)} & q \to 0 \\ F(q) &\sim \sqrt{-\log\left(2\pi (1-q)^2\right) - \log\left(-\log\left(2\pi (1-q)^2\right)\right)} & q \to 1 \end{aligned}$$

First we derive the asymptotic expressions for strategies. These follow from the asymptotic behavior of $F'(q)^{-1} = \sqrt{1/(2\pi)}e^{-\frac{1}{2}F(q)^2}$. Substituting the above approximations for $F(q)^2$ we get:

$$F'(q)^{-1} \sim 2q\sqrt{-\pi \log q} \quad q \to 0$$

$$F'(q)^{-1} \sim 2(1-q)\sqrt{-\pi \log(1-q)} \quad q \to 1$$

which yield the resulting approximations for strategies when combined with Lemma **??**.

We have shown in the proof of Step Lemma **??** that that $V_0'(q) = q\Delta'(q) = q(2r)^{-1/2}F'(q)$, which combined with the above expression for $F'(q)^{-1}$ yields the given asymptotic approximation for$V_0'(q)$ as $q \to 0$.

Since $F'(q)^{-1} \to 0$ as $q \to 1$, we have $\sqrt{2r}V_0(q) \sim qF(q) \sim F(q)$ as $q \to 1$, which yields the given expression for $V_0(q)$ as $q \to 1$.

As $V(q)$ is a constant times $F'(q)^{-1}$ the asymptotic behavior of V follows directly from the asymptotic behavior of $F'(q)^{-1}$.

A.5 Expected Hitting Times

PRELIMINARIES We use the tools developed in Karlin and Taylor (1981) Chapter 15 parts 3 and 6. WLOG we restrict attention to state $\theta = 0$ and the true belief $q = 0$. Following section 15.3 in Karlin and Taylor (1981), define the scale functions:

$$s_0(x) = e^{-2\int^x \frac{\mu_0(q)dq}{\varsigma^2(q)}} = e^{2\int^x \frac{dq}{1-q}} = (1-x)^{-2} \quad \text{and} \quad S_0(x) = \int^x \frac{dy}{(1-y)^2} = \frac{1}{1-x}.$$

Define the variable hitting time $\mathcal{T}(q_0, q)$ as time elapsed before the belief first hits q starting at q_0. For any $0 < q_0 < 1$ define $\mathcal{T}_0(q_0) \equiv \lim_{q\to 0} \mathcal{T}(q_0, q)$ and $\mathcal{T}_1 \equiv \lim_{q\to 1} T(q_0, q)$.

The expected time until the belief q leaves any interval $[a, c] \subset (0, 1)$ is $\propto r^{-1}$. As the process governing the evolution of beliefs is linear in r^{-1}, so are all hitting time solutions.

Beliefs converge to the truth in the limit. In state $\theta = 0$, there is a positive chance that beliefs $q < a$ converge to 0 before hitting a.

As shown in Karlin and Taylor (1981), if $Pr(T_0(q, 0) < T(q, c)) > 0$ for $0 < q < c < 1$ and $Pr(T_1(q, 0) < T(q, a)) = 0$ for any $0 < a < q < 1$ then the uninformed player learns the true state eventually. Lemma 6.1 in Karlin and Taylor (1981) establishes that $S_0(b) - \lim_{a \to 0} S_0(a) < \infty$ for some $0 < c < 1$ implies $Pr(T_0(q, 0) < T(q, c)) > 0$. Trivially, this is satisfied in our case. Lemma 6.1 also establishes that $\lim_{b \to 1} S_0(b) - S_0(a) = \infty$ for some $0 < a < 1$ implies $Pr(T_1(q, 0) < T(q, a)) = 0$, which is again trivially satisfied.

Equation (3.10) on page 195 provides a formula for the probability that the belief hits upper barrier c before lower barrier a, $u_\theta(q, a, c)$. In the low state we have $u_0(q, 0, c) = [q(1-c)]/[c(1-q)]$, while the calculations for $u_0(q, a, 1) = (q-a)/[q(1-a)]$ are symmetric.

The expected time until the truth is revealed is not finite. In section 15.6 Karlin and Taylor (1981) establish (Lemma 6.2) that the true state will not be revealed in finite time (i.e. $E[T(q, 0)|L] = \infty$) if there exists $0 < c < 1$ such that $\Sigma(0, c) = \infty$, where:

$$\Sigma(0, c) \equiv \lim_{a \to 0} \int_a^c \left[\int_y^c [q^2(1-q)^2(n_1(q) - n_0(q))^2 s_0(q)]^{-1} dq \right] s_0(y) dy.$$

Since Σ differs between the constrained and unconstrained models, we must evaluate each separately.

UNCONSTRAINED Σ: Substituting for $n_1 - n_0$ and s_0 we have:

$$\begin{aligned}
\Sigma(0,c) &= \lim_{a\to 0}\int_a^c \left[\int_y^c (1-q)^2\Delta'(q)^2 dq\right](1-y)^{-2}dy \\
&\geq \lim_{a\to 0}\int_a^c \left[\int_y^c \Delta'(q)^2 dq\right]\frac{(1-c)^2}{(1-y)^2}dy \\
&\geq \lim_{a\to 0}\int_a^c \left[\int_y^c \Delta'(q)^2 dq\right](1-c)^2 dy \\
&= \frac{\pi(1-c^2)}{r}\lim_{a\to 0}\int_a^c \left[\int_y^c e^{2\,\mathrm{erf}^{-1}(2q-1)^2}dq\right]dy \\
&= \frac{\pi(1-c)^2}{2r}\lim_{a\to 0}\int_a^c \left[\mathrm{erfi}(\mathrm{erf}^{-1}(2c-1)) - \mathrm{erfi}(\mathrm{erf}^{-1}(2y-1))\right]dy \\
&= \frac{\pi(1-c)^2}{2r}\left(\mathrm{erfi}(\mathrm{erf}^{-1}(2c-1))b - \lim_{a\to 0}\int_a^c \left[\mathrm{erfi}(\mathrm{erf}^{-1}(2y-1))\right]dy\right)
\end{aligned}$$

Since $0 < c < 1$ the lead term will be finite. So, $\Sigma(0,c) = \infty$ *iff* the following is infinite:

$$\begin{aligned}
&-\lim_{a\to 0}\int_a^c \mathrm{erfi}(\mathrm{erf}^{-1}(2y-1))dy \\
&= k + \lim_{a\to 0}\mathrm{erf}^{-1}(2a-1)^2\sum_{n=0}^{\infty}\frac{a_n(-\mathrm{erf}^{-1}(2a-1)^2)^n}{n!}
\end{aligned}$$

for some finite k, $a_{n+1}/a_n = (n+1)^2/[(n+3/2)(n+2)]$, and $a_0 = 1$. Since $\lim_{z\to -1}\mathrm{erf}^{-1}(z)^2 = \infty$ the limit on the RHS can be written:

$$\lim_{y\to\infty} y\sum_{n=0}^{\infty}\frac{a_n(-y)^n}{n!} = \lim_{y\to\infty} y\left[y + \sum_{n=1}^{\infty}\left(\frac{a_{2n}y^{2n+1}}{(2n)!} - \frac{a_{2n-1}y^{2n}}{(2n-1)!}\right)\right].$$

Thus, the limit is infinite since we have:

$$\sum_{n=1}^{\infty}\frac{a_{2n}(2n-1)!}{a_{2n-1}(2n)!} = \sum_{n=1}^{\infty}\frac{(2n+1)^2}{(2n+3/2)(2n+2)2n} = \infty.$$

CONSTRAINED Σ: Since we must only evaluate the limit $\Sigma(0,c)$ for some $c > 0$, we choose c low enough such that partial mixing obtains, for then $n_1(q) - n_0(q) = (1-q)^{-1}$

and we may explicitly evaluate the given integrals.

$$
\begin{aligned}
\Sigma(0,c) &= \lim_{a\to 0}\int_a^c \left[\int_y^c \frac{(1-q)^2}{q^2}dq\right]\frac{dy}{(1-y)^2} \\
&= \lim_{a\to 0}\int_a^c \left[\frac{c^2-1}{c}+\frac{1-y^2}{y}+2\log\left(\frac{y}{c}\right)\right]\frac{dy}{(1-y)^2} \\
&= \lim_{a\to 0}\frac{(c-a)(1+c)+(1+a)c\log(a/c)}{(a-1)c} = \infty
\end{aligned}
$$

$\square$

A.6 Time and Money in the Unconstrained Model

We seek an equation linking time and money. Specifically, the expected time until the information rent $V(q)$ first falls below some threshold $\bar{V}$ given starting value V. The information rent is symmetric about $q = 1/2$, rising below $1/2$ and falling above $1/2$. If we fix symmetric barriers $(a, 1-a)$ then $T(q)$, the expected time to first hitting one of these barriers given starting $q \in (a, 1-a)$, is also symmetric about $q = 1/2$ and co-monotonic with $V(q)$. Thus, we may define the expected time as a function of value: $h(V)$. Toward characterizing h, apply the chain rule to get:

$$
h'(V(q)) = T'(q)/V'(q)
$$

and again to get:

$$
h''(V(q)) = \frac{T''(q) - h'(V(q))V''(q)}{V'(q)^2} \tag{20}
$$

To an ODE for h in V we must express V', V'', and T'' as a function of V.

To eliminate T'', rearrange the $T(q)$ equation from (9) as follows:

$$
-1 = \frac{2r}{F'(q)^2}T''(q) \quad \Rightarrow \quad T''(q) = -\frac{1}{r}F'(q)^2. \tag{21}
$$

Our solution to the unconstrained model is

$$
V(q) = \frac{1}{\sqrt{2r}F'(q)} \quad \Rightarrow \quad F'(q)^2 = \frac{1}{2rV(q)^2}.
$$

Thus, we may substitute out for $F'(q)^2$ and combine with the differential equation for

T (21) to get $T''(q)$ as a function of V:

$$T''(q) = -\frac{1}{2rrV(q)^2}. \tag{22}$$

Now use that $F(q)$ is the Probit function, with known derivative:

$$F'(q) = 2\sqrt{\pi}e^{\frac{1}{2}F(q)^2} \quad \Rightarrow \quad V(q) = \frac{1}{2\sqrt{2r\pi}}e^{-\frac{1}{2}F(q)^2} \tag{23}$$

and thus:

$$V'(q) = -\frac{1}{\sqrt{2r}}F(q) \quad \Rightarrow V'(q)^2 = \frac{1}{2r}F(q)^2 \tag{24}$$

Log both sides of the right equation in (23) and rearrange to get:

$$\tfrac{1}{2}F(q)^2 = -\log(2\sqrt{2r\pi}V(q)) \equiv -\log(kV(q)).$$

Now combine this with Equation (24) to express $V'(q)$ as a function of $V(q)$:

$$V'(q)^2 = -\tfrac{1}{r}\log(kV(q)). \tag{25}$$

Differentiate the $V'(q)$ equation (24) to get:

$$V''(q) = -\frac{1}{\sqrt{2r}}F'(q) \quad \Rightarrow \quad -\frac{1}{2rV''(q)} = \frac{1}{\sqrt{2r}F'(q)}.$$

combining this with the solution to our intensity model $V(q) = [\sqrt{2r}F'(q)]^{-1}$ we get:

$$V''(q) = -\frac{1}{2rV(q)} \tag{26}$$

Now substitute for T'', V', and V'' from equations (22), (25), and (26) in equation (20) to get:

$$h''(V) = \frac{1}{2rV^2}\left[\frac{1 - rVh'(V)}{\log(kV)}\right]$$

The general solution to this differential equation is:

$$h'(V) = \frac{c}{\sqrt{-\log(kV)}} + \frac{k\sqrt{\pi}\ \mathrm{erfi}(\sqrt{-\log(kV)})}{2r\sqrt{-\log(kV)}},$$

We now argue that the constant c must be zero by working with a boundary condition at $h'(V(1/2))$. Since $h'(V(q)) = T'(q)/V'(q)$ and both $T'(1/2) = 0$ and $V'(1/2) = 0$ we must work with the limit as $q \to 1/2$:

$$h'(V(1/2)) = \lim_{q\to 1/2} \frac{T'(q)}{V'(q)} = \lim_{q\to 1/2} \frac{T''(q)}{V''(q)} = \frac{-F'(q)^2/r}{-F'(q)/\sqrt{2r}} = \lim_{q\to 1/2} \frac{\sqrt{2}}{\sigma\sqrt{r}} F'(q) = \frac{2\sqrt{2\pi}}{\sigma\sqrt{r}} = \frac{k}{r}.$$

Since $F(1/2) = 0$, $V(1/2) = (2\sqrt{2r\pi})^{-1} = k^{-1}$. Altogether we have boundary condition:

$$h'(1/k) = \frac{k}{r}.$$

Then we evaluate

$$\lim_{V\to 1/k} \frac{k\sqrt{\pi}\ \mathrm{erfi}(\sqrt{-\log(kV)})}{2r\sqrt{-\log(kV)}} = \frac{k}{r},$$

which implies $c = 0$. Then we must have:

$$h(V) - \hat{c} = \int^V h'(s)ds$$

for some constant $\hat{c}$. Toward solving for this integral, define the monotonic transformations:

$$Z(V) \equiv \sqrt{-\log(kV)} \quad \text{and} \quad z(s) \equiv \sqrt{-\log(ks)}$$

and thus:

$$V = \tfrac{1}{k} e^{-Z^2} \quad \text{and} \quad ds = -\frac{2z}{k} e^{-z^2} dz.$$

So that we have:

$$\begin{aligned}
h(V) - \hat{c} &= \int^V h'(s)ds = -\frac{k\sqrt{\pi}}{2r} \int^{Z(V)} \frac{\mathrm{erfi}(z)}{z} \left(\frac{2z}{k} e^{-z^2}\right) dz \\
&= -\frac{\sqrt{\pi}}{r} \int^{Z(V)} \mathrm{erfi}(z) e^{-z^2} dz = -r^{-1} Z(V)^2 F_{(2|2)}(1, 1; \tfrac{3}{2}, 2; -Z(V)^2) \\
&= r^{-1} \log(kV) F_{(2|2)}(1, 1; \tfrac{3}{2}, 2; \log(kV)),
\end{aligned}$$

where $F_{2|2}(1, 1; \frac{3}{2}, 2; y)$ is the generalized hypergeometric function that admits representation:

$$F_{2|2}(1, 1; \tfrac{3}{2}, 2; y) = \sum_{n=0}^{\infty} \frac{a_n y^n}{n!},$$

$a_{n+1}/a_n = (n+1)^2/[(n+3/2)(n+2)]$, and $a_0 = 1$.

Now substitute define $\nu = kV$ and $\tau(\nu) = h(V/k)$ to get the given $\tau(\nu)$ and $\tau'(\nu)$ expressions. Differentiating the later yields the given $\tau''(\nu)$ expression. To see that $\tau''(\nu)$ is negative, define the transformation

$$z(\nu) = \sqrt{-\log z} \Rightarrow v = e^{-z^2}.$$

The we can write the numerator of $\tau''(\nu)$ as $z\sqrt{\pi}e^{-z^2}\ \mathrm{erfi}(z) - 2z^2$, which is negative *iff:*

$$\mathrm{erfi}(z) \leq \frac{2z}{\sqrt{\pi}}e^{z^2}.$$

To see that this must hold, note that $z \geq 0$, both sides of the inequality evaluate to 0 at $z = 0$, and the right hand side increases more steeply than the left hand side as:

$$\begin{aligned} \mathrm{erfi}'(z) &= \frac{2}{\sqrt{\pi}}e^{z^2} \\ \frac{\partial\left(\frac{2z}{\sqrt{\pi}}e^{z^2}\right)}{\partial z} &= \frac{2}{\pi}\left(e^{z^2} + 2z^2e^{z^2}\right). \end{aligned}$$

□

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