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Peer group effects in applied general equilibrium★

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Summary. In this paper, I develop an applied general equilibrium environment with peer group effects. The application I consider is schooling. The framework used here is general equilibrium with clubs. I establish the existence of equilibrium for the economy with a finite number of school types. This result is then extended to the case where the set of school types is a continuum. The two welfare theorems are shown to hold for both economies. To compute the equilibrium, I construct a Negishi mapping from the set of weights on individual type's utility to the set of transfers that support the corresponding Pareto allocations as competitive equilibria with transfers. Because this mapping is a correspondence, a version of Scarf's algorithm is used to find a competitive equilibrium.

Keywords and Phrases: Education, Peer effects, Applied general equilibrium, Clubs.

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1 Introduction

In the standard theory of production the technology set is modeled as a relationship between feasible combinations of inputs and outputs. In this type of framework the organizational structure is abstracted from. Often times this approach is sufficient as the organization of inputs is not important to the production process. However, for some issues organization of the production inputs does matter. For example, if there are peer effects in schooling, how students are arranged across

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schools will affect educational outcomes, and perhaps future earnings. Another important organization is the family. The way in which families form can affect savings and demography. There can also be peer effects at the firm level. People may care about their work environment, and, for instance, prefer to work with cooperative rather than non cooperative people. The organization of these firms will then affect productivity and hence output. In all of these cases, an approach other than the standard is needed to address organizational issues. The purpose of this paper is to develop a framework that can be used to predict equilibrium outcomes when organization matters. It is then a simple exercise to study how interventions that constrain organizational activities affect equilibrium outcomes.

I develop an applied general equilibrium framework that can be used to predict private school formation and composition, when there are peer group effects. I establish that an equilibrium exists and that the two welfare theorems hold. I take a constructive approach in the proof of the Second Welfare Theorem and propose a candidate price system that can support any Pareto optimal allocation as a competitive equilibrium with transfers. Using this direct approach, I do not merely establish the existence of a supporting price system, I find one. These prices can then be used in the computation of a competitive equilibrium, making applied general equilibrium models with peer group effects practical.

To compute an equilibrium, I construct a Negishi (1960) mapping from the set of weights on individual types from the social planner's problem to the set of supporting transfers. I have shown that standard computational methods, such as Newton, are not always successful, because the Negishi mapping is a correspondence. This correspondence arises from the linearity of the utility function over schooling. A more general search algorithm, which can deal with correspondences, is needed. The method proposed here is based on Scarf (1973).

The framework used here is general equilibrium with clubs. To endogenize club membership I follow Cole and Prescott (1997) and permit randomizing across schools. By doing so, I do not, ex ante, rule out any trades, and therefore allow for a complete set of markets. Introducing the lottery simplifies the analysis by giving rise to convex preferences. Scotchmer (1994) and Ellickson, Grodal, Scotchmer, and Zame (1997), confront the same problem of endogenizing club membership using a different approach. They do not allow randomizing across club membership. In their model, consumers choose full membership in one or more clubs. In the case of a finite number of agents and a finite number of possible clubs, they develop a notion of approximate equilibrium which enables them to deal with the "integer" problem that plagues club economies. This problem disappears in the case of a continuum of agents, Ellickson, Grodal, Scotchmer, and Zame (1998).

Given this environment and computational method, it is a simple exercise to study how interventions that constrain organizational activities affect equilibrium outcomes. In order to predict how students will sort across schools under a specific policy, the set of possible school types need only be restricted to exclude those schools which do not meet the criteria of that policy, see Caucutt (forthcoming). Many of the ideas formalized here were first developed in Rothschild and White (1995).[1] Other relevant work that considers the effects of education policy when there are peer effects in schooling includes, Epple and Romano (1998), De Bartolome (1990), Benabou (1996), and Nechyba (1996).

The paper proceeds as follows. In Section 2, I lay out the the basic structure of the model, and define an equilibrium. In Section 3, existence of an equilibrium is established, and in Section 4, the welfare theorems are shown to hold. In Section 5, I construct the Negishi mapping used in the computation and I outline the computational method.

2 Basic structure

In this section, I specify the model of endogenous school formation. The framework is general equilibrium with clubs and builds on Cole and Prescott (1997). The key feature of the model is the technologies that are available to groups of individuals to jointly produce human capital. The human capital that an individual acquires depends on three factors, his type, or ability to learn, the per student input of resources, or expenditures at his school, and the relative numbers of the various types of students attending his school, or the student body composition of his school. This last factor is the peer group. A school type is characterized by its per student expenditures and its student body composition.

Because it simplifies the analysis, I consider schools that have been normalized to one student, and I permit the number of schools of a given type to vary. I can do this because schooling displays constant returns to scale. An alternative and equivalent approach is to have one school of each type, and allow its scale to vary.

There is a finite number of school types that a child can attend. This is generalized in the next section. If the parent were to choose the school type her child attended, preferences would not be convex, since school membership would be discrete. Consequently, there could be mutually beneficial gambles, as in Rogerson (1988). Therefore, a parent instead chooses the probability that her child attends each school type. This convexifies preferences and ensures that all gains from trade are exhausted.

2.1 The environment

There are I types of parents, endowed with varying levels of human capital. For simplicity a one-to-one relationship between human capital and income is assumed. Each parent has one child, who is born with some learning ability, and this information is public. Students with higher learning ability, all else constant, get more out of schooling than children with lower learning ability. The initial endowments of human capital and learning ability, are given by $h^i,\ i = 1, \ldots, I$, and $a^i,\ i = 1, \ldots, I$, respectively. There is a continuum of type i parents of

[1] I thank an anonymous referee for bringing this paper to my attention.

measure $\lambda^i > 0$, and $\sum_i \lambda^i = 1$. So parents differ over their income, and their children's learning ability. A parent cares about consumption and the human capital that her child receives.

There is a finite number of school types. Because the choice to attend a specific school is exclusionary, in the sense that attending one school precludes a student from attending another, a randomizing mechanism is introduced to convexify the problems facing the parents. With convexity, attention can be and is restricted to type identical allocations. This is illustrated in Section 2.3. Therefore, a parent maximizes expected utility by choosing the probability that her child attends each school type.

2.2 The economy

Commodity space

Because I assume utility is additively separable, the commodity space is $L = R^{1+S^I}$, where S is the number of possible schools a child can attend. An element of the commodity space is denoted by $x = (c, \pi^1, \pi^2, ..., \pi^I)$, where c is personal consumption, and the π^i are vectors, $(\pi^i_1, \pi^i_2, ..., \pi^i_S)$, of probabilities associated with each possible school. The consumption possibilities set for type i is,

$$X^i = \{x \in L_+ : \sum_s \pi^i_s \leq 1, \pi^j_s = 0,\ \forall j \neq i \text{ and } \forall s\}.$$

Consumption is non-negative and the probabilities, π^i_s satisfy, $0 \leq \pi^i_s \leq 1,\ \forall i, s$. A type i parent has $\pi^j = 0,\ \forall j \neq i$.[2] Because $\sum_s \pi^i_s \leq 1$, a parent can choose not to send her child to school.

Preferences

Utility, $U^i : X^i \rightarrow R$ is defined as,

$$U^i(x) = u(c) + \sum_s v^i_s \pi^i_s,$$

where u is unbounded, differentiable, strictly concave, monotone, and $u'(c) \rightarrow \infty$ as $c \rightarrow 0$. Here, v_s is the utility a parent receives if her child attends school s. This utility is a function of the human capital the child acquires from school s, and is therefore exogenous. It is also bounded. A parent receives utility from the consumption good, and from the type of school her child attends. The parent's endowment is given by $\omega^i = (h^i, \mathbf{0}, \mathbf{0}, ..., \mathbf{0}) \in \mathbf{L}_+$. The parent is endowed only with a positive amount of the consumption good.

[2] Suppose there are four types of schools and two types of parents. A type 1 parent has perhaps, $\pi^1 = (.5, 0, .25, .25)$, and $\pi^2 = (0, 0, 0, 0)$. While a type 2 parent has, $\pi^1 = (0, 0, 0, 0)$, and perhaps, $\pi^2 = (0, .5, .5, 0)$. It is necessary to have both vectors of probabilities included in the commodity space, because the probability a type 1 student attends school 1 is a different commodity than the probability a type 2 student attends school 1, and therefore will be priced differently.

Technologies

Let $\mathscr{F}$ be the set of possible school types. Each school s in $\mathscr{F}$, is defined by the fraction of each type in attendance, n_s^i, $i = 1, ..., I$, and its per pupil expenditures, e_s. Recall a school is normalized to a size of one student. Let z be a vector containing the measures of each school type, z_s. The set of possible school types, $\mathscr{F} \in R^{1+I}$, is exogenous, while the measure of each school type, z, is endogenous. The aggregate production possibility set is:

$$Y = \{y \in L : \exists z \ s.t. \sum_s e_s z_s + y_c \leq 0;\ y_s^i = z_s n_s^i, \forall i, s\}.$$

An element of the aggregate production possibility set is a vector $y = (y_c, y^1, y^2, ..., y^I)$. The amount of consumption inputed to schooling is given by $y_c \leq -\sum_s e_s z_s$. The vector $y^i = (y_1^i, y_2^i, ..., y_S^i)$ is a vector containing the measures of type i children attending each school in $\mathscr{F}$, $y_s^i = z_s n_s^i$, $\forall s$.

Resource constraint

The resource constraint is $\sum_i \lambda^i (x^i - \omega^i) = y$. This implies that $\pi_s^i = \frac{n_s^i z_s}{\lambda^i}$, $\forall s, i$, and $\sum_i \lambda^i c^i + \sum_s z_s e_s \leq \sum_i \lambda^i h^i$. The first $I \cdot S$ constraints ensure that the probabilities the parent chooses match the measures the schools choose. The last constraint is the consumption resource constraint.

2.3 Competitive equilibrium

Prices

The price system is a vector $p \in L$. The value of a commodity point, x, is $p \cdot x$. Let c be the numeraire. Therefore, prices are in terms of the consumption good. Since the output good is the numeraire, the resource input to schooling is simply referred to as per student expenditures.

Definition of equilibrium

A *type identical competitive equilibrium* is a price system p and an allocation $[\{x^{i*}\}, y^*]$ such that:

1. x^{i*} maximizes $U^i(x)$ subject to $x^i \in X^i$ and $p \cdot (x^i - \omega^i) \leq 0$, $i = 1, ...I$,
2. y^* maximizes $p \cdot y$ subject to $y \in Y$, and
3. $\sum_i \lambda^i (x^i - \omega^i) = y$.

Type identical allocations

As mentioned previously, I restrict my attention to type identical allocations. If a competitive equilibrium exists with individuals of a given type consuming different allocations, then there is another competitive equilibrium with the same price system, p, production, y, and utilities, U^i, such that everyone of a given type consumes identical allocations. This identical allocation is the average allocation across individuals of a given type. These are the type identical equilibria which I consider. Note that if a competitive equilibrium exists with individuals of a given type consuming different allocations, then each of these allocations must only differ across the probabilities, π_s, and not across consumption c. This is because the utility over schooling is linear, while the utility over consumption is strictly concave.

Proposition 2.3.1. *If there is an equilibrium with x differing across individuals of the same type, then an equilibrium exists such that:*

$$x^i = \overline{x}^i \equiv \frac{1}{\lambda^i} \int x^i(j)dj, \ \forall i,$$

with p and y the same.

Proof. This follows directly from the linearity of utility over schooling.

Q.E.D.

3 Existence

In this section existence of equilibrium is established. This is done first for the case considered in the previous section, where $\mathscr{F}$ is finite. The result is then extended to include cases where $\mathscr{F}$ is no longer finite, but is still a compact subset of R^{1+I}.

3.1 Existence of an equilibrium when $\mathscr{F}$ is finite

I begin by establishing existence when $\mathscr{F}$ is finite. The randomizing device used to convexify the commodity space, gives rise to some slight modifications of the standard existence proof. This is due to the fact that most of the commodities are probabilities, which are often zero and always bounded from above by one.

Proposition 3.1.1. *A competitive equilibrium exists in this environment.*

Proof. If the following conditions hold, then a competitive equilibrium exists (McKenzie (1981)).

C1. The consumption sets, X^i, are closed and convex.
C2. The preference ordering is continuous and quasi-concave.
C3. Y is a closed convex cone.

C4. $Y \bigcap X^i + \{-\omega^i\}$, is not empty for all i.
C5. $Y \bigcap L_+ = \{0\}$.
C6. There is a common point in the relative interiors of Y and $X = \sum_i \lambda^i(X^i + \{-\omega^i\})$.
C7. For any two nonempty, disjoint partitions of I, I_1 and I_2, and for $x_{I_1} = y - x_{I_2}$, where $x_{I_1} \in \sum_{i \in I_1} \lambda^i(X^i + \{-\omega^i\})$, $x_{I_2} \in \sum_{i \in I_2} \lambda^i(X^i + \{-\omega^i\})$, and $y \in Y$, there exists a $v \in \sum_{i \in I_2} \lambda^i(X^i + \{-\omega^i\})$ such that, $x'_{I_1} + x_{I_2} + v \in Y$, and x'_{I_1} can be decomposed into an allocation for I_1 that is weakly preferred by all members of I_1 and is strictly preferred by at least one member of I_1.

Conditions, C1, C2, and C3 clearly hold. Condition C4 follows from the fact that a person can always consume her endowment. If the commodity space does not include commodities which produce something from nothing, condition C5 holds. Since all schools require a strictly positive input of expenditures, no commodity can be costlessly produced. Condition C6 is established by the following argument. Let x^i be such that, $c = \epsilon/\lambda^i$, $\pi^i = \epsilon/\lambda^i$, and $\pi^j = 0$, $\forall i, j \neq i$. For any small enough ϵ, $x^i \in X^i$, hence $y = \sum \lambda^i(x^i - \omega^i)$ is in the relative interior of $\sum \lambda^i(X^i - \{\omega^i\})$. Note, to be in Y, $y^i_s = \lambda^i \pi^i_s$, and $y_c \leq -\sum_s e_s \lambda^i \pi^i_s / n^i_s$. The first requirement is satisfied because it implies $y^i_s = \epsilon$. The consumption resource constraint implies that $y_c \geq I\epsilon - \sum_i \lambda^i h^i$. The second requirement is then satisfied for sufficiently small ϵ, because $I\epsilon + \sum_s e_s \epsilon / n^i_s \leq \sum_i \lambda^i h^i$, for sufficiently small ϵ. Thus y is in the relative interior of Y. Condition C7 ensures that all parents have positive income. This is satisfied because in any feasible allocation, more consumption is preferred to less.

Q.E.D.

3.2 Existence of an equilibrium when there is a continuum of school types

The set of possible school types, $\mathscr{F}$, is no longer restricted to be finite, it is, however, a compact subset of R^{1+I}. Let $\Pi = M(\mathscr{F})$ be the space of signed measures on the Borel-sigma algebra of $\mathscr{F}$. The implicit topology on $\mathscr{F}$ is induced by the Euclidean metric. The space of signed measures contains the space of probability measures. An element $\pi^i \in \Pi$ is a probability measure over $\mathscr{F}$. The commodity space is now the space of consumption crossed with the space of signed measures I times, where I is the number of types of students. Therefore, $L = R \times \Pi^I$. The first component of the commodity space is personal consumption. The price system is a continuous linear functional $p : L \to R$, which has representation, $p \cdot x = c + \int p(s)\pi(ds)$.

I begin this section by sketching the argument for the existence of equilibrium when there is a continuum of school types. I first show that an equilibrium exists if the aggregate technology set is restricted so that only a finite number of school types is permitted to operate. Note that the commodity space is still the general commodity space, $L = R \times \Pi^I$, and the price system p is still a continuous linear functional mapping L into R. These economies are referred to as r^{th} approximate economies, where r denotes the number of school types allowed to operate. Next, I show a convergent subsequence of these equilibria exists and that the limit is an equilibrium for the economy with no restrictions on the production set. Again, the commodity space and price space in the restricted and unrestricted economies are the same.

For the r^{th} approximate economy, the aggregate production set is restricted to those elements whose support belongs to the finite set $\mathscr{F}_r \subseteq \mathscr{F}$, where all points in $\mathscr{F}$ lie in a $1/r$-neighborhood of at least one point in $\mathscr{F}_r$. The r^{th} finite economy of Section 2, is the economy where the commodity space L_r is finite dimensional. It has been shown, Proposition 3.1.1, that an equilibrium exists for the finite economy. Any competitive equilibrium in a finite dimensional economy corresponds to an equilibrium in the approximate economy, with the general commodity space and the restricted production set. The measures of the approximate economy put mass on those schools that have positive probability in the finite economy equilibrium, and are zero elsewhere.

I then show that there exists a convergent subsequence such that the limit is an equilibrium for the economy with the unrestricted production set Y. To do this, I establish that there exists a subsequence for which the allocations converge weak star and the continuous price function converges uniformly. The uniform convergence of the subsequence of continuous price functions ensures that the limiting price function is continuous. Such a subsequence also ensures that the limits of the sequences x_r^i are optimal, given the price system $p = \lim_{r\to\infty} p_r$, where r indexes elements of a subsequence for which allocations converge weak star and prices converge uniformly.

This subsection is outlined in the following way. I start by proving the price functions, p_r, are bounded uniformly in r. Because the price functions are functionally related to the Lagrange multiplier on the probability constraint in the consumer's problem, it suffices to show that this Lagrange multiplier is bounded. I use the fact that prices are bounded from below to establish the Lagrange multiplier is bounded from above. Since prices are bounded uniformly, and, in addition, the Lipschitz condition is shown to hold, there exists a subsequence for which the continuous price function converges uniformly. I then show that the allocations converge weak star. This follows from the fact that the consumption sets are closed and bounded, and therefore weak star compact. Lastly, I demonstrate that the limiting allocation and price system satisfy the requirements for a competitive equilibrium.

Lemma 3.2.1. *Prices, p, are bounded from below by $\overline{p}$, where $\overline{p} \geq h - u^{-1}(\overline{U} - \max_s v_s)$.*

Proof. Recall that $u(c)$ is unbounded, differentiable, and strictly concave. Let F be the set of feasible allocations. F is closed and bounded. Let $\overline{U} = \max_{x\in F} U$. A competitive equilibrium allocation cannot yield higher utility than $\overline{U}$. If p were sufficiently negative we would have $U(p) > \overline{U}$, which is a contradiction.

Find $\overline{p}$ such that $U = \overline{U}$:

$$\bar{U} \geq u(h - \bar{p}) + \max_s v_s.$$

Solving for $\bar{p}$, yields,

$$\bar{p} \geq h - u^{-1}(\bar{U} - \max_s v_s).$$

Q.E.D.

Lemma 3.2.2. *The Lagrange multiplier on the probability constraint in the consumer's problem, ϕ_r, is uniformly bounded in r.*

Proof.

i. The Lagrange multiplier, ϕ_r is bounded from below, $\phi_r \geq 0$. If a parent is given a little extra probability to allocate across schools, ϕ_r is the resulting change in the objective. Because the probability constraint is not a binding constraint, a parent can always choose not to use the extra probability. Therefore, extra probability cannot make a parent worse off, and $\phi_r \geq 0$.
ii. The Lagrange multiplier, ϕ_r is bounded from above. An increase in probability of size δ, can increase the objective by at most:

$$\delta \max_s v_s - [u(c) - u(c - \delta\bar{p})].$$

This is equivalent to:

$$\delta \max_s v_s - \delta\bar{p}u'(c).$$

Therefore,

$$\phi_r^{max} = \delta \max_s v_s - \delta\bar{p}u'(c),$$

where $\bar{p}$, $\max_s v_s$, and $u'(c)$ are all bounded.[3]
Hence, ϕ_r is bounded from above.

Q.E.D.

Lemma 3.2.3. *The price functions, p_r, are bounded uniformly in r.*

Proof. From the first order conditions of the consumer's problem,

$$p_r(s) \geq \frac{v_s - \phi_r}{u'(c)}, \quad \forall s$$

with equality if s belongs to the support of x, where ϕ_r is the Lagrange multiplier on the probability constraint. We can choose prices so that, $p_r(s) = \frac{v_s - \phi_r}{u'(c)}$, $\forall s$. Since ϕ_r, v_s, and $u'(c)$ are bounded, p_r must be bounded as well.

Q.E.D.

[3] Given that as $c \to 0$, $u'(c) \to \infty$, the c that solves the consumer's problem will always be strictly positive, therefore the corresponding $u'(c)$ is bounded.

Note that prices are only uniquely determined at operating schools. When a school does not operate, I choose the smallest price that is consistent with zero demand. The Lagrange multiplier, ϕ_r, can be thought of as the net benefit from schooling to the parent. For any operating school, s, it is just the difference between the utility received from attending that school, υ_s, minus the cost in terms of consumption lost, $p_r(s)u'(c)$.

Lemma 3.2.4. *The sequence of price functions, $\{p_r\}$, converge uniformly.*

Proof. The Lipschitz condition, $\exists k > 0$ s.t. $\forall s, s' \in \mathscr{F}$, $|p_r(s) - p_r(s')| \leq kd(s, s')$, and uniform boundedness are sufficient conditions for uniform convergence. Lemma 3.2.3 established that p_r is bounded uniformly.
Using the price function,

$$|p_r(s) - p_r(s')| = |\frac{\upsilon_s - \upsilon_{s'}}{u'(c)}|.$$

Since, υ_s and $u'(c)$ are bounded, clearly,

$$|\frac{\upsilon_s - \upsilon_{s'}}{u'(c)}| \leq kd(s, s').$$

Therefore, the Lipschitz condition holds and we have uniform convergence.

Q.E.D.

Definition. A sequence $\{x_r\}$ converges *weak star* to x^*, if for all $f \in C(\mathscr{F})$,

$$\int f(\mathscr{F}) x_r(d\mathscr{F}) \to \int f(\mathscr{F}) x^*(d\mathscr{F}),$$

where $C(\mathscr{F})$ is the set of continuous functions on $\mathscr{F}$, a compact metric.

Lemma 3.2.5. *For any sequence $\{x_r\}$, $x_r \in X$, there is a subsequence which converges weak star.*

Proof. Because X is a closed and bounded subset of L, X is weak star compact. Therefore, for any sequence $\{x_r\}$, $x_r \in X$, there is a subsequence which converges weak star.

Q.E.D.

Lemma 3.2.6. *For any feasible sequence $\{y_r\}$, there is a subsequence which converges weak star.*

Proof. To be feasible it must be the case that:

$$y_r = \sum_i \lambda^i (x_r^i - \omega^i).$$

Since x_r^i converges weak star, y_r converges weak star as well.

Q.E.D.

The next Lemma uses the uniform convergence of prices and the weak star convergence of allocations to establish that $\lim_{r\to\infty}\int p_r(s)\pi_r(ds) = \int \lim_{r\to\infty} p_r(s)\lim_{r\to\infty}\pi_r(ds)$. This result is used several times in the proof that the convergent subsequences converge to a competitive equilibrium in the limiting economy.

Lemma 3.2.7. *For any $\epsilon > 0$, $\exists N$, such that $\forall s \in \mathscr{F}$ and $\forall r \geq N$ we have $|\int p_r(s)\pi_r(ds) - \int p(s)\pi(ds)| < \epsilon$.*

Proof. By adding and subtracting $\int p(s)\pi_r(ds)$ we have the following equality,

$$|\int p_r(s)\pi_r(ds) - \int p(s)\pi(ds)| = |\int p_r(s)\pi_r(ds) \\ - \int p(s)\pi_r(ds) + \int p(s)\pi_r(ds) - \int p(s)\pi(ds)|.$$

Using the triangle inequality, and the linearity of p, we have:

$$|\int p_r(s)\pi_r(ds) - \int p(s)\pi(ds)| \leq |\int p_r(s)\pi_r(ds) \\ - \int p(s)\pi_r(ds)| + |\int p(s)[\pi_r(ds) - \pi(ds)]|.$$

From the weak star convergence of x we know that, $|\int p(s)[\pi_r(ds) - \pi(ds)]| \to 0$, as $r \to \infty$. And from the uniform convergence of p we know that, $|\int p_r(s)\pi_r(ds) - \int p(s)\pi_r(ds)| < \epsilon$. Therefore, $|\int p_r(s)\pi_r(ds) - \int p(s)\pi(ds)| < \epsilon$.

Q.E.D.

Proposition 3.2.1. *A competitive equilibrium exists when there is a continuum of school types.*

Proof.

i. Prices and allocations converge, $p_r \to p^*$, $x_r \to x^*$, and $y_r \to y^*$.
   I have shown that p_r converges uniformly, and that allocations converge weak star.
ii. Allocations, x^* and y^*, satisfy the resource constraint, $\sum_i \lambda^i(x^{*i} - \omega^i) = y^*$.
   This follows from the fact that allocations converge weak star.
iii. Allocations, x^* and y^*, are optimal, given, p^*.
   a. Household optimality
      Given p_r, $[\{x_r\}, y_r]$ is a sequence of equilibria converging to $[\{x^*\}, y^*]$. It needs to be shown that given p^*, x^* is an equilibrium allocation for the household. This entails demonstrating two things, first, given p^*, x^* is feasible, and second, there is no other allocation $\tilde{x}$ such that $U(\tilde{x}) > U(x^*)$.
      Because x_r is an equilibrium allocation given p_r, x_r is feasible:

$$c_r + \int p_r(s)\pi_r(ds) \leq h,$$

$$\int \pi_r(ds) \leq 1.$$

Allocations converge weak star so it follows that,

$$\int \pi^*(ds) \leq 1.$$

It was established in Lemma 3.2.7 that $\int p_r(s)\pi_r(ds) \to \int p^*(s)\pi^*(ds)$. Together with the fact that allocations converge weak star, this implies:

$$c^* + \int p^*(s)\pi^*(ds) \leq h.$$

Therefore, x^* is feasible.
Because x_r is an equilibrium allocation given p_r, it is optimal:

$$U(x_r) \geq U(\tilde{x}),$$

$\forall \tilde{x}$ such that $\int \tilde{\pi}(ds) \leq 1$, and $\tilde{c} + \int p_r(s)\tilde{\pi}(ds) \leq h$.
Given Lemma 3.2.7 and the fact that allocations converge weak star, it follows that:

$$U(x^*) \geq U(\tilde{x}),$$

$\forall \tilde{x}$ such that $\int \tilde{\pi}(ds) \leq 1$, and $\tilde{c} + \int p^*(s)\tilde{\pi}(ds) \leq h$.
Therefore, x^* is optimal.

b. School optimality
It needs to be shown that given p^*, y^* is an equilibrium allocation. This is done in two parts. First, it is established that y^* is feasible. Second, it is shown that y^* is optimal.
Because y_r is an equilibrium allocation given p_r, it is feasible:

$$y_{r,c} \leq \int e(s)z_r(ds),$$

$$y^i(s) = z_r(s)n^i(s), \forall i, s.$$

Since allocations converge weak star we have,

$$y_c^* \leq \int e(s)z^*(ds),$$

$$y^{i*}(s) = z^*(s)n^i(s), \forall i, s.$$

Therefore, y^* is feasible.
Because y_r is an equilibrium allocation given p_r, it is optimal:

$$\int p_r(s)y_r(ds) \geq \int p_r(s)\tilde{y}(ds),$$

$\forall \tilde{y}$ such that $\tilde{y}_c \leq \int e(s)\tilde{z}(ds)$, and $\tilde{y}^i(s) = \tilde{z}(s)n^i(s)$.
Given Lemma 3.2.7, which holds for allocations y as well as x, and from the weak star convergence of the allocation, it follows that:

$$\int p^*(s)y^*(ds) \geq \int p^*(s)\tilde{y}(ds),$$

$\forall \tilde{y}$ such that $\tilde{y}_c \leq \int e(s)\tilde{z}(ds)$, and $\tilde{y}^i(s) = \tilde{z}(s)n^i(s)$.
Therefore, y^* is optimal.

Q.E.D.

4 Welfare Theorems

This section establishes the two welfare theorems. The first part contains the First Welfare Theorem. The proof does not depend upon the dimensionality of the commodity space. Therefore, it holds, not only for the case of a finite number of school types, but also for the case of a continuum of school types. The second part is devoted to the Second Welfare Theorem, and it is divided into two sections. I begin by proving that the result holds when the set of school types is finite. I then extend the result to economies with a continuum of school types.

4.1 The First Welfare Theorem

Proposition 4.1.1. *Every competitive equilibrium allocation is Pareto optimal.*

Proof. Preferences are convex, and no type is satiated. Since I is finite, the value of the aggregate endowment is finite. Therefore, Debreu's (1954) proof of the first welfare theorem holds. It is infeasible to satiate with a type identical allocation. The convexity of preferences and non satiation guarantees local non satiation at all points in the intersection of the consumption set and the budget constraint.

Q.E.D.

4.2 The Second Welfare Theorem

4.2.1 When $\mathscr{F}$ is finite

I show that in this framework, there exists a price system, p, that supports each Pareto optimal allocation as a quasi-competitive equilibrium with transfers. I then verify that there is a cheaper point in the consumption possibility set. This implies that the quasi-competitive equilibrium is in fact a competitive equilibrium.

The following five conditions are used to prove that a price system exists that supports each Pareto allocation as a quasi-competitive equilibrium. All five conditions hold in this environment.

A1. For each i, the consumption set X^i is convex.
A2. For each i, if $x, x' \in X^i$, $U^i(x) > U^i(x')$, and $\theta \in (0,1)$ then $U^i[\theta x + (1-\theta)x'] > U^i(x')$.

A3. For each i, U^i is continuous.
A4. The aggregate production possibility set, Y, is convex.
A5. The commodity space, L, is finite dimensional.

Proposition 4.2.1. *Let conditions A1-A5 hold, let $[\{\hat{x}\}, \hat{y}]$ be a Pareto optimal allocation, and assume that for some person $i \in 1, ...I$, there is an $\overline{x}^i \in X^i$ with $U^i(\overline{x}) > U^i(\hat{x})$. Then there exists a price system, p, such that,*

i. for each i, $x^i \in X^i$ and $U^i(x) > U^i(\hat{x})$ implies $p \cdot (x^i - \omega^i) \geq p \cdot (\hat{x}^i - \omega^i)$;
ii. $y \in Y$ implies $p \cdot y \leq p \cdot \hat{y}$.

Proof. Stokey and Lucas, with Prescott (1989, p. 455). Note that the conditions used here are slightly stronger than those of Debreu (1954).

Q.E.D.

The following proposition establishes that there is a price system such that any Pareto optimal allocation can be supported as not only a quasi-competitive equilibrium with transfers, but as a competitive equilibrium with transfers.

Proposition 4.2.2. *There exist prices, p, that support each Pareto optimal allocation as a competitive equilibrium with transfers, where transfers are given by, $t^i = p(\omega^i - x^i)$, $\forall i$.*

Proof. From Stokey and Lucas, with Prescott (1989, p. 456), all that needs to be shown is that for each person there is a point, in the consumption possibility set, which is cheaper than the Pareto optimal allocation.

This condition holds here, because everyone will always be consuming positive amounts $c > 0$, so that $c = 0$ (which is in the consumption possibility set) will be cheaper, holding z_s fixed.

Q.E.D.

4.2.2 When $\mathscr{F}$ is a continuum

The standard method of proving the Second Welfare Theorem, when the commodity space is not finite dimensional, relies on the existence of an interior point in the production set. Because of the lottery over schooling, such a condition fails to hold in this environment. I, instead, take a constructive approach, proposing a candidate price system, and showing that it supports any Pareto optimal allocation as a competitive equilibrium with transfers. By using a direct approach to prove the Second Welfare Theorem, I not only show that supporting prices exist, I also construct them. These prices can then be used in the computation of competitive equilibrium, making applied general equilibrium models with peer effects practical.

The social planner's problem

I begin by showing that given any set of weights on the utilities of individual types in the social planner's problem, the solutions to the planner's problem are Pareto optimal. I then show that for any Pareto optimal allocation, there exist a set of weights in the planner's problem for which that allocation is the solution. In other words, I can use the solutions corresponding to the weighted social planner's problem to represent the set of Pareto optimal allocations. This is outlined in Lemmas 4.2.2 and 4.2.3.

The following problem is the $\theta-$weighted social planner's problem, which is used in the next three lemmas. The Lagrange multipliers on the i probability constraints of this problem, are denoted by $\mu^i(\theta)$.

$$\begin{array}{ll} \max_{z,c} & \sum_i \theta^i \lambda^i [u(c^i) + \int v^i(s) n^i(s) z(ds)/\lambda^i] \\ \text{s.t.} & \sum_i \lambda^i c^i + \int e(s) z(ds) = \sum_i \lambda^i h^i, \\ & \int n^i(s) z(ds)/\lambda^i \leq 1, \quad \forall i. \end{array} \tag{1}$$

Lemma 4.2.1. *As long as the constraint set is nonempty, there exists a solution* $\{\hat{z}(\theta), \hat{c}(\theta)\}$ *to the* θ*-planner's problem.*

Proof. The lemma follows from the continuity of the objective function and the compactness of the constraint set. The constraint set is compact because it is bounded, closed, and finite dimensional. Consumption and expenditures on schooling are bounded by the resource constraint, and the probabilities are bounded by zero and one. The constraint set is closed due to the weak inequalities that define the set.

Q.E.D.

Lemma 4.2.2. *Given a Pareto optimal allocation* $[\{\hat{x}\}, \hat{y}]$, *there exists a vector of welfare weights* $\hat{\theta}$, *nonnegative and not all zero, such that* $[\{\hat{x}\}, \hat{y}]$ *is a solution to the* $\hat{\theta}-$*social planner's problem.*

Proof. The utility possibility set is nonempty, closed, and convex, due to the convexity of the X^i and Y, and the concavity of the U^i. Given any point, $[\{\hat{x}\}, \hat{y}]$, on the boundary of the utility possibility set, there exists at least one supporting hyperplane to the utility possibility set passing through $[\{\hat{x}\}, \hat{y}]$. The slope of the hyperplane corresponds to a vector of welfare weights $\hat{\theta}$, nonnegative and not all zero.

Q.E.D.

Lemma 4.2.3. *For any strictly positive vector of welfare weights* $\hat{\theta}$, *the solution* $[\{\hat{x}\}, \hat{y}]$ *to the* $\hat{\theta}-$*social planner's problem is Pareto optimal.*

Proof. This follows from the social planner's problem.

Q.E.D.

Therefore, the set of Pareto optimal allocations and the set of solutions to social planner's problems are the same, when $\theta^i > 0, \ \forall i$. And we can safely exclude the cases where some $\theta^i = 0$, because these allocations would never correspond to a competitive equilibrium (everyone must not be worse off than under autarky). Hence, the weighted social planner's problem can be used to represent the set of Pareto optimal allocations, when constructing a mapping to search for competitive equilibria.

A constructive approach to the second welfare theorem

I take a constructive approach to proving the Second Welfare Theorem. I propose a candidate price system, and then show that if an allocation is Pareto optimal and the prices are given by the proposed candidate price system, then both consumers and schools are optimizing. Therefore, the allocation can be supported as a competitive equilibrium with transfers. The next proposition establishes this Second Welfare Theorem by verifying that the first order conditions, which are necessary and sufficient for consumer and school maximization, are satisfied given the candidate price system, $\hat{p}$, and the Pareto optimal allocation $[\{\hat{x}\}, \hat{y}]$. These first order conditions are necessary and sufficient because utility is concave, and the constraints are linear. The consumer's problem is written with transfers included:

$$\begin{array}{ll} \max_{\pi,c} & u(c^i) + \int v^i(s)\pi^i(ds) \\ \text{s.t.} & c^i + \int p^i(s)\pi^i(ds) = h^i - t^i, \\ & \int \pi^i(ds) \leq 1. \end{array} \tag{2}$$

The school's problem is to choose how many of each type of school to operate to maximize profits.

$$\max_z \quad \int [\textstyle\sum_i p^i(s)n^i(s) - e(s)] z(ds). \tag{3}$$

Proposition 4.2.2. *Given any Pareto optimal allocation* $[\{\hat{x}\}, \hat{y}]$,*the price system,*

$$\hat{p}^i(s) = \frac{v^i(s) - \hat{\mu}^i/(\hat{\theta}^i\lambda^i)}{u'(\hat{c}^i)},$$

supports $[\{\hat{x}\}, \hat{y}]$ *as a competitive equilibrium with transfers,* t*, where, by Lemma 4.2.2,* $\hat{\theta}$ *is the set of weights that support* $[\{\hat{x}\}, \hat{y}]$ *as a solution to the social planner's problem (1), and* $\hat{\mu}$ *is the resulting set of Lagrange multipliers on the probability constraints in that problem.*

Proof.

i. First, I show that the consumers are optimizing.
   The allocation x solves the household's problem (2) given p, if and only if for some ϕ the following three necessary and sufficient first order conditions hold,

$$p^i(s) \geq \frac{v^i(s) - \phi^i}{u'(c^i)}, \ \forall s,$$

with equality if s belongs to the support of x,

$$c^i + \int p^i(s)\pi(ds) \leq h^i - t^i,$$

$$\int \pi(ds) \leq 1.$$

To establish the first condition, I use the candidate price system, $\hat{p}$, and define an appropriate $\hat{\phi}^i$. The candidate price system is given by:

$$\hat{p}^i(s) = \frac{v^i(s) - \hat{\mu}^i/(\hat{\theta}^i\lambda^i)}{u'(\hat{c}^i)}.$$

If, $\hat{\phi}^i \equiv \hat{\mu}^i/(\hat{\theta}^i\lambda^i)$, the candidate price system can be written as:

$$\hat{p}^i(s) = \frac{v^i(s) - \hat{\phi}^i}{u'(\hat{c}^i)}.$$

Consequently, given $p = \hat{p}$, $x = \hat{x}$, and the appropriately chosen $\phi = \hat{\phi}$, the first condition holds. The second condition holds trivially. The third condition follows from the fact that $\hat{x}$ is Pareto optimal.

ii. Second, I show that the school is optimizing. The following first order conditions are necessary and sufficient for school optimization (3):

$$\sum_i n^i(s)p^i(s) - e(s) \leq 0, \quad \forall s.$$

It holds with equality if s belongs to the support of z. All that needs to be established is that:

$$e(s) \geq \sum_i n^i(s)p^i(s), \quad \forall s,$$

holding with equality when s belongs to the support of z. Since the allocation is Pareto optimal, it is a solution to a weighted social planner's problem (1), and therefore satisfies the following first order conditions:

$$e(s) \geq \sum_i n^i(s)[\frac{v^i(s) - \hat{\mu}^i/(\hat{\theta}^i\lambda^i)}{u'(\hat{c}^i)}], \quad \forall s,$$

holding with equality when s belongs to the support of z. Substitute in the candidate price system:

$$e(s) \geq \sum_i n^i(s)\hat{p}^i(s), \quad \forall s,$$

holding with equality if s belongs to the support of z. Therefore the school is optimizing.

Q.E.D.

5 Computing an equilibrium

In this section I discuss the computational subtleties encountered when trying to find a competitive equilibrium in this environment. I construct a mapping from the set of weights on the utilities of individual types in a social planner's problem to the set of transfers that support the corresponding Pareto optimal allocations as competitive equilibria with transfers. This is referred to as the Negishi mapping. Given a vector of weights, I begin by finding a Pareto optimal allocation. This is a straightforward computation based on the weighted social planner's problem. I then take the price system proposed in Proposition 4.2.2, and calculate the transfers necessary to support the Pareto allocation as a competitive equilibrium with transfers. This completes the mapping from the space of individual weights to the space of transfers. Unfortunately this mapping is a correspondence. Consequently, standard Newton methods applied to this mapping may fail to converge to a competitive equilibrium. In order to consistently find a competitive equilibrium, a method of search that can be applied to correspondences is needed. I use a grid search algorithm based on Scarf (1973) to find a competitive equilibrium.

5.1 The Negishi mapping

The Negishi method of finding a competitive equilibrium is based on a mapping from a vector of individual weights in a social planner's problem to a vector of transfers that support the corresponding Pareto optimal allocations as competitive equilibria.

To find a Pareto allocation computationally, I rewrite the planner's problem by moving the consumption resource constraint into the objective. There is now a finite number of schools. Let the Lagrange multiplier on the consumption resource constraint be Γ.

$$\begin{array}{ll} \max_{z,c} & \sum_i \theta^i \lambda^i u(c^i) - \Gamma(\sum_i \lambda^i (c^i - h^i)) + \\ & \quad \sum_s z_s[(\sum_i \theta^i n_s^i v_s^i) - \Gamma e_s] \\ \text{s.t.} & \sum_s z_s n_s^i / \lambda^i \leq 1, \ \forall i. \end{array} \tag{4}$$

The first order condition with respect to c yields, $\theta^i \lambda^i u'(c^i) - \Gamma \lambda^i = 0$. Therefore, the c^i, $\forall i$ follow from Γ. So, given Γ, the problem is linear in z, and can easily be solved for z using linear programming techniques.[4]

In Proposition 4.2.2 I propose a price system that supports each Pareto optimal allocation as a competitive equilibrium with transfers. These prices depend

[4] The algorithm I use to find a Pareto allocation, given θ, begins with choosing the Lagrange multiplier on the consumption resource constraint, Γ. Given Γ and θ, individual consumption, c^i, immediately follows from the first order condition with respect to individual consumption. I then solve the planner's problem (4) for z using a linear program. I next check if the resource constraint holds. If not, I adjust the Lagrange multiplier, using a bisection method, and repeat the process, until the consumption resource constraint is satisfied. This allocation is a θ-Pareto allocation.

upon the Lagrange multipliers from the planner's problem. Solving the planner's problem using linear programming techniques yields these multipliers. Transfers are then,

$$t^i(\theta) = h^i - c^i(\theta) - \sum_s p_s^i(\theta)\frac{z_s(\theta)n_s^i}{\lambda^i}, \ i = 1, \ldots, I.$$

A mapping has been constructed from the space of individual weights to the space of transfers. The last step of this computational procedure is to search over the space of weights for a vector of weights which results in a zero transfer vector. This vector of weights gives rise to a Pareto optimal allocation that can be supported as a competitive equilibrium with no transfers. This allocation, along with the corresponding price system, is a competitive equilibrium.

5.2 Complications

Unfortunately, the Negishi mapping is a correspondence. This can cause a standard Newton algorithm to fail to converge. The correspondence arises because for some θ-weights, the planner's problem (1) does not have a unique solution. Total consumption is the same for all equilibria that correspond to the same θ-weight, and this together with the relationship between θ and individual consumption, implies that individual consumption is also the same. Total resources used in schooling must also be the same for all equilibria resulting from the same θ-weight. However, the way in which these resources are allocated varies. Therefore, any θ-weight that yields a continuum of equilibria corresponds to a continuum of associated transfers.

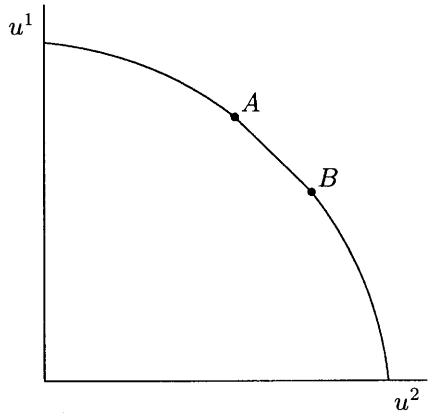

Figure 1

Imagine a utility space, as in Figure 1, with two types of people. Given θ, a θ-Pareto allocation is the point of tangency between a line with slope θ, and the utility possibility frontier. Notice that the utility possibility frontier is linear between points A and B. For some θ-weights, all convex combinations of

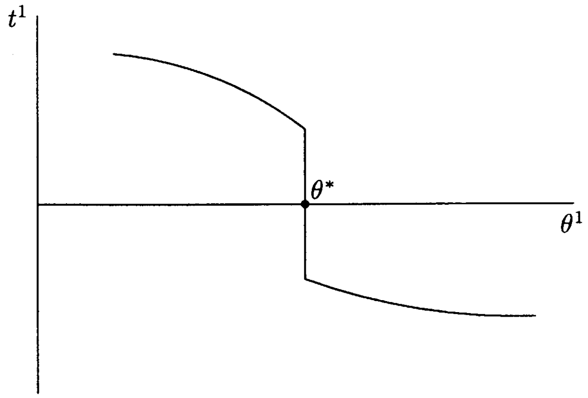

Figure 2

A and B are Pareto optimal allocations. Suppose one of those allocations is a competitive equilibrium (see Fig. 2). As the Newton algorithm converges from one side, transfers to type 1 are negative and converging to what they are at A. Coming from the other side, transfers to type 1 are positive and converging to their value at B. The algorithm will oscillate from side to side, failing to converge to an equilibrium. If a competitive equilibria does not fall in the linear region, Newton will converge (Fig. 3).

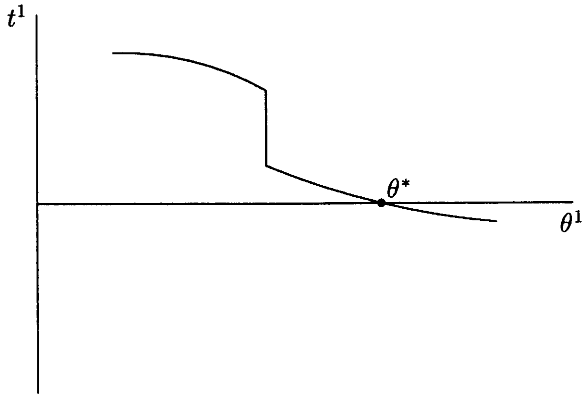

Figure 3

5.3 Scarf's Algorithm

The method I use to find a competitive equilibrium is an algorithm based on Scarf (1973). This algorithm is general enough to find an equilibrium when the mapping is a correspondence. It uses the Negishi mapping of Section 5.1, and searches over the simplex of weights for a vector of weights that gives rise to a Pareto allocation that can be supported as a competitive equilibrium with no transfers. The idea is to search in such a way that only a small fraction of the grid must be considered. The algorithm starts with a subsimplex and searches by removing one vertex and replacing it with another. This process continues until an approximate competitive equilibrium is found.

This algorithm can be thought of as a constructive alternative to the existence proof in Section 3. In the limit, Scarf's algorithm yields a competitive equilibrium. Suppose a sequence of grids is chosen so that each consecutive grid becomes finer. For each grid, the algorithm yields a subsimplex. As the grid becomes dense, a subsequence of these subsimplices can be found such that the subsimplices converge to a single vector $\hat{\theta}$.

Proposition 5.3.1. *$\hat{\theta}$ is a competitive equilibrium.*

The proof of Proposition 5.3.1 can be found in Appendix A. The idea is that there is sequence of approximate equilibria associated with making the grid finer. This sequence has a convergent subsequence, and the limit of the convergent subsequence is an equilibrium.

Computationally, Scarf's algorithm finds a set of social planner's weights which imply a set of equilibrium conditions which are approximately the "true" conditions. It cannot be said that these social planner weights are close to the "true" equilibrium (see Fig. 4). Appendix A contains the details of Scarf's algorithm.

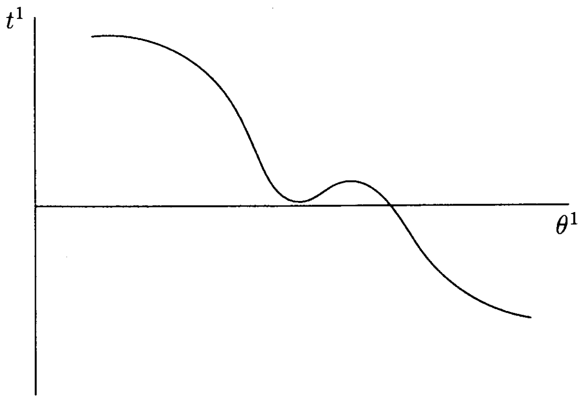

Figure 4

A Scarf's algorithm

A.1 The Algorithm

Imagine there are three types of people. The weight simplex would be as in Figure 5. If the simplex is divided into subsimplices as shown, the grid is then

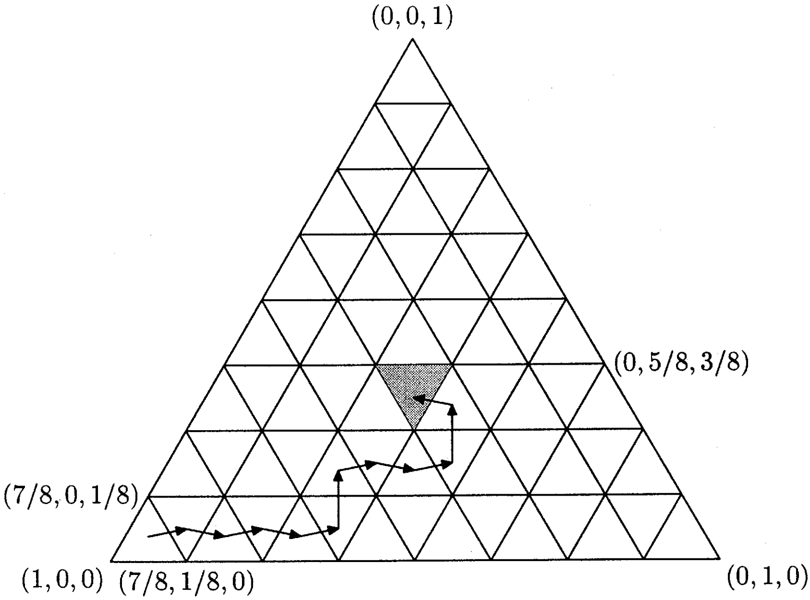

Figure 5

made up of points of the form $(a_1/8, a_2/8, a_3/8)$, where a_1, a_2, and a_3 are nonnegative integers that sum to 8. There is necessarily a subsimplex where any convex combination of its vertices is an approximate competitive equilibrium, and Scarf's algorithm gives a method for finding this subsimplex in an efficient manner.

Let there be an $I-$dimensional simplex corresponding to I types of people. Each vertex in the simplex is given a vector label, $L(\theta^i) = t^i + e$, where $t^i \in T(\theta^i)$ and e is a vector of ones. Here T is the Negishi correspondence. The labels for the sides of the simplex, $(\theta^1, ..., \theta^I)$, are as follows:

$$\begin{array}{ccc} \theta^1 & \to & (1, 0, ..., 0) \\ \theta^2 & \to & (0, 1, ..., 0) \\ . & & . \\ . & & . \\ \theta^I & \to & (0, 0, ..., 1). \end{array} \tag{5}$$

Construct a matrix A containing all of the label vectors:

$$A = \begin{bmatrix} 1 & 0 & . & . & . & 0 & L_{1,I+1} & . & . & . & L_{1,k} \\ 0 & 1 & . & . & . & 0 & L_{2,I+1} & . & . & . & L_{2,k} \\ . & . & & & & . & . & & & & . \\ . & . & & & & . & . & & & & . \\ 0 & 0 & . & . & . & 1 & L_{I,I+1} & . & . & . & L_{I,k} \end{bmatrix}$$

The first I columns correspond to the labels for the weights on the sides of the simplex, $(\theta^1, ..., \theta^I)$, and the last $I+1$ through k columns correspond to the labels for the weights in the simplex, $(\theta^{I+1}, ..., \theta^k)$.

Given a vector $e = (1, ..., 1)$ of length I, the vector of interest is $\alpha = (\alpha_1, ..., \alpha_k)$, with $\alpha_i \geq 0, \ \forall i$, such that $A\alpha = e$. Scarf's algorithm makes use of the fact that the set of non-negative solutions to $A\alpha = e$ is bounded implies that there exists a subsimplex $(\theta^j, ..., \theta^{j+I})$ such that the columns j through $j+I$ of A form a feasible basis for $A\alpha = e$. The idea is that a subsimplex of I vectors can be chosen from $(\theta^1, ..., \theta^k)$, such that the vector e is contained in the convex hull of their images.

The system of equations, $A\alpha = e$, can be written in the following form:

$$\begin{array}{ccccccc} \alpha_1 & & & + & \sum_{j=I+1}^k \alpha_j(t_1^j + 1) & = & 1 \\ & \alpha_2 & & + & \sum_{j=I+1}^k \alpha_j(t_2^j + 1) & = & 1 \\ & & \cdot & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot \\ & & \alpha_I & + & \sum_{j=I+1}^k \alpha_j(t_I^j + 1) & = & 1, \end{array} \tag{6}$$

with $\alpha_i \geq 0$. The $\alpha_i, \ i = 1, ..., I$, are slack variables, which are zero if the i^{th} side of the simplex is not a member of the subsimplex in question.

In the limit Scarf's algorithm yields a competitive equilibrium. Suppose a sequence of grids is chosen so that each consecutive grid becomes finer and finer. For each grid, the algorithm yields a subsimplex, I columns of A, and the I $\alpha-$weights associated with this subsimplex. As the grid becomes dense, a subsequence of these subsimplices can be found such that:

1. The subsimplices converge to a single vector $\hat{\theta}$.
2. Each column, j, of A, corresponding to a nonslack vector and having a positive weight, converges to a vector, $(t_1^j+1, t_2^j+1, ..., t_I^j+1)'$. By the upper semicontinuity of the mapping, T, each $\hat{t}^j = (\hat{t}_1^j, ..., \hat{t}_I^j)$ is contained in $T(\hat{\theta})$.
3. The positive weights of $A\alpha = e$ converge to a sequence of non-negative weights $\hat{\alpha}$.

Proposition 5.3.1 *$\hat{\theta}$ is a competitive equilibrium.*

Proof.
It suffices to show that $0 \in T(\hat{\theta})$. First, rewrite system 6:

$$\begin{array}{ccccccc} \hat{\alpha}_1 & & & + & \sum_l \hat{\alpha}_{j_l}(\hat{t}_1^{j_l} + 1) & = & 1 \\ & \hat{\alpha}_2 & & + & \sum_l \hat{\alpha}_{j_l}(\hat{t}_2^{j_l} + 1) & = & 1 \\ & & \cdot & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot \\ & & \cdot & & \cdot & & \cdot \\ & & \hat{\alpha}_I & + & \sum_l \hat{\alpha}_{j_l}(\hat{t}_I^{j_l} + 1) & = & 1, \end{array} \tag{7}$$

with $\hat{\alpha}_i \geq 0$, and equal to zero if $\hat{\theta}_i > 0, \ i = 1, ..., I; \ \hat{t}^{j_l} \in T(\hat{\theta})$; and $\hat{\alpha}_{j_l} \geq 0$.

I will show that $\hat{\alpha}_1 = \hat{\alpha}_2 = ... = \hat{\alpha}_I = 0$, $\sum_l \hat{\alpha}_{j_l} = 1$, and hence that $0 \in T(\hat{\theta})$.

I begin by showing that $\sum_l \hat{\alpha}_{j_l} \geq 1$. Take the equations from 7 with $\hat{\theta}_i > 0$, and hence $\hat{\alpha}_i = 0$. Suppose that there are N such equations. Add these N equations:

$$\sum_{\hat{\theta}^i > 0} \sum_l \hat{\alpha}_{j_l} (\hat{t}_i^{j_l} + 1) = N.$$

Interchange the summation:

$$\sum_l \hat{\alpha}_{j_l} \sum_{\hat{\theta}^i > 0} \hat{t}_i^{j_l} + N \sum_l \hat{\alpha}_{j_l} = N.$$

Note that $\sum_{\hat{\theta}^i > 0} \hat{t}_i^{j_l} \leq 0$. This follows from the fact that total transfers must sum to zero, and that here the sum is excluding transfers to only those people who have zero weight put upon them, and hence have transfers which are non-negative. Therefore,

$$N \sum_l \hat{\alpha}_{j_l} \geq N,$$

and,

$$\sum_l \hat{\alpha}_{j_l} \geq 1.$$

I next show that $\sum_l \hat{\alpha}_{j_l} = 1$, and that $\hat{\alpha}_1 = \hat{\alpha}_2 = ... = \hat{\alpha}_I = 0$.

Sum over all of the equations of the system 7,

$$\sum_i \hat{\alpha}_i + \sum_i \sum_l \hat{\alpha}_{j_l} (\hat{t}_i^{j_l} + 1) = I.$$

Recall that $\sum_i \hat{t}_i^{j_l} = 0$, hence,

$$\sum_i \hat{\alpha}_i + I \sum_l \hat{\alpha}_{j_l} = I.$$

It has been established that, $\sum_l \hat{\alpha}_{j_l} \geq 1$. Therefore, $\sum_i \hat{\alpha}_i \leq 0$. Since $\hat{\alpha}_i \geq 0$, $\forall i$, it follows that,

$$\hat{\alpha}_1 = \hat{\alpha}_2 = ... = \hat{\alpha}_I = 0.$$

And, consequently,

$$\sum_l \hat{\alpha}_{j_l} = 1.$$

Given these two conditions, system 7 reduces to:

$$
\begin{aligned}
\textstyle\sum_l \hat{\alpha}_{j_l} \hat{t}_1^{j_l} &= 0 \\
\textstyle\sum_l \hat{\alpha}_{j_l} \hat{t}_2^{j_l} &= 0 \\
&\vdots \\
\textstyle\sum_l \hat{\alpha}_{j_l} \hat{t}_I^{j_l} &= 0\,.
\end{aligned}
\tag{8}
$$

Since $\hat{t}^{j_l} \in T(\hat{\theta})$ by construction, and $T(\hat{\theta})$ is convex,

$$0 \in T(\hat{\theta}).$$

Therefore, $\hat{\theta}$ is the vector of weights corresponding to a competitive equilibrium.

Q.E.D.

Since the algorithm can never actually reach $\hat{\theta}$, it is necessary to approximate a competitive equilibrium. Therefore, let

$$\hat{\theta} = \sum \alpha_i \theta^i / \sum \alpha_i\,,$$

be an approximate competitive equilibrium. The nature of this approximation is discussed in the next section.

It is important to understand how the algorithm moves through the simplex. The subsimplex which can be used to approximate a competitive equilibrium is reached through a series of linear programming pivot steps. The algorithm begins with a subsimplex made up of the vectors $(\theta^2, ..., \theta^I, \theta^j)$, and a feasible basis consisting of the labels $(L^1, ..., L^I)$. The label L^j is brought into the feasible basis and a pivot step is performed on this new column. A unique column in the feasible basis is then eliminated. The corresponding vector in the subsimplex is removed and a new vector is introduced to the subsimplex. This process continues until either L^1 is removed from the feasible set or θ^1 is introduced to the subsimplex, which implies that the columns of the subsimplex and the feasible basis coincide. The proof that a pivot step can always be carried out and the conditions under which the vector to be eliminated is unique are contained in Chapter 4 (Scarf, 1973), along with the proof that a subsimplex exists with labels that form a feasible basis.

Scarf's algorithm must converge to an approximate competitive equilibrium in a finite number of steps. This is due to the fact that the grid is finite and that the algorithm cannot cycle (Scarf, 1973, pp. 45–48). One weakness of Scarf's algorithm is that the closeness of the approximation depends on the size of the grid, but the finer the grid, the slower the algorithm. However, there have been advances in this area (Arrow and Kehoe, 1992). One such advance is Merrill's (1971) algorithm.

Scarf's algorithm requires that the starting vertex be a corner of the simplex. Merrill's algorithm allows the initial starting point to be any vertex within the simplex. The algorithm begins with a coarse grid and an initial starting point. An approximate competitive equilibrium is found, the grid is made finer, and the process begins again with the approximate competitive equilibrium as the next initial guess. This continues until the desired accurateness of the approximate competitive equilibrium is reached.

A.2 Approximation

Suppose the algorithm, with grid size n, gives rise to the simplex, $(\theta^{n1}, \theta^{n2}, ...)$ and associated transfers $(t^{n1}, t^{n2}, ...)$, such that, $t^{nj} \in T(\theta^{nj})$. As discussed in the previous section, the approximate equilibrium is chosen to be,

$$\hat{\theta} = \sum_j \alpha^{nj} \theta^{nj} / \sum \alpha^{nj},$$

$$\hat{t} = \sum_j \alpha^{nj} t^{nj} / \sum \alpha^{nj}.$$

where the α's are the weights defined in the system of equations 6.

Proposition A.2.1 formalizes the nature of the approximation. This proposition depends upon T being upper semicontinuous.

Proposition A.2.1. (Scarf, 1973, pp. 90–93) *Let δ be an arbitrary positive number. Then there is an $\epsilon > 0$, with the following property: Take the grid so fine that any vectors in the subsimplex corresponding to a primitive set have a distance less than or equal to ϵ.*

Then there is a vector θ^*, with $|\hat{\theta} - \theta^*| \leq \delta$, such that $\hat{t}$ has a distance less than or equal to δ from $T(\theta^*)$.

Basically this proposition is saying that given any $\delta > 0$, there is an ϵ size grid, such that the equilibrium conditions of the approximate competitive equilibrium, $\hat{t}$, are within δ distance of the "true" equilibrium conditions, $T(\theta^*)$, which in this case is a vector of zeros. So, I can choose a threshold level of error, δ, and find a grid fine enough to yield an approximate competitive equilibrium with transfers less than δ. As pointed out in Figure 4, this does not imply that the approximate competitive equilibrium is within δ of a "true" equilibrium, but only that the equilibrium conditions are within δ of the "true" equilibrium conditions.

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