# Companion (online) Appendix to: "The Nature of Credit Constraints and Human Capital"

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## 1 Introduction

This document is a companion (online) appendix to our paper *"The Nature of Credit Constraints and Human Capital."* Here, we consider three important extensions of the two-period model used in the paper and show that our main analytical results are robust to these generalizations.

1. **Endogenous parental transfers:** In the paper, we take individual/student wealth $w$ as fixed and study the effects of changes in ability; we also consider the implications of changing $w$. In section 2 of this appendix, we allow for the endogenous determination of an individual's wealth $w \geq 0$ from parental bequests. We consider the effects of ability on investment holding parental wealth constant, as well as the effects of changing parental wealth.

2. **Ability affecting the cost to invest.** The paper implicitly assumes that the marginal cost of investment is the same for everyone. In Section 3 of this appendix, we allow this marginal cost to depend on individual ability.

3. **Generalized earnings function.** In Section 4 of this appendix, we generalize the earnings function to have non-unitary elasticity of substitution between ability $a$ and human capital $h$. Here, we generalize our results about the ability – investment relationship, providing sufficient conditions in terms of the consumption intertemporal elasticity of substitution (IES) and the elasticity of substitution between ability and investment in human capital production.

For these extensions, we briefly discuss the alternative models and focus on whether they alter our main analytical results in the paper. Overall, we show that our main results are quite robust. While at a theoretical level, it is possible to define investment cost functions or human capital production functions that significantly alter certain results in our paper, these cases are extreme and empirically implausible. The implicit assumptions in our paper are common and largely consistent with available evidence. We conclude that there is no evidence that warrants the drastic departures from our assumptions required to overturn our main theoretical conclusions.

# 2 Endogenous Parental Transfers

We now show that our main analytical results in the paper extend to an intergenerational context where parents endogenously decide how much wealth to transfer to their children. In the paper, we present a number of comparative statics results on the relationship between investment and ability. Those results are *conditional on $w$*, i.e. they *hold the young individual's wealth $w$ constant*. In this section, we obtain equivalent results that hold the individual's family wealth $w_p$ constant and let $w$ be determined endogenously. We also discuss the relationship between $w_p$ and investment.

We consider three standard models of intergenerational transfers: "altruistic", "warm-glow" and "paternalistic" parents. They differ in their assumptions about parental preferences. In the first model, parents care about the *utility* of their child. In the second, parents directly care about the *amount they transfer* to their child. In the third model parents care about the amount of *human capital investment* of their child. For transparency and simplicity, we consider each of these three models in a two-generation model like that of Section 4 of the paper. Here, the parent is old and only lives during the first period. The child is young, lives two periods, invests in human capital in the first and earns income in the second.

We briefly summarize our results now and provide some detailed derivations in the rest of this section.

1. Under "altruistic" preferences, all of our qualitative results in Section 4 of the paper hold with the sole re-interpretation of initial wealth as parental wealth.

2. Under "warm-glow" preferences, all of our results in Section 4 hold interpreting wealth as either parental transfers or parental wealth.

3. Under "paternalistic" preferences (with a few additional conditions), all of our Section 4 results hold for exogenous constraints and endogenous constraints under limited commitment. They also hold for the GSL under a slight modification of the tied-to-investment constraint.

## 2.1 Altruistic preferences

Consider the standard altruistic model in which the utility of the parent, $U^p$, depends on his own consumption, $c_p$, and the utility of his child, $U^c$. The utility of the child depends on his current consumption, $c_0$, and consumption in the next period, $c_1$. Preferences are $U^P = u(c_p) + \theta U^c$ and $U^c = u(c_0) + \beta u(c_1)$, which follow standard one-sided altruistic models. The notation is the same as in our paper except for the new parameter $\theta \in [0,1]$ defining parental altruism. The parent and the child observe the child's ability, $a$, which impacts his future earnings as in the paper. Parents allocate their wealth, $w_p > 0$, between their own consumption and transfers to their children, $b \geq 0$. Given transfers, the child chooses consumption in both periods, human capital investment, and borrowing so as to maximize his own utility. For expositional purposes, we assume that the child has no initial wealth of his own (i.e. $w = b$); although, this assumption can easily be relaxed.

This problem can clearly be formulated as a Pareto optimal allocation problem (with weights 1 and $\theta$ on the parent and child, respectively), where the planner chooses $c_p$, $c_0$, $c_1$, $d$, and $h$. We now consider allocations under different forms of constraints.

**Unrestricted Allocations** The intergenerational problem with exogenous constraints is

$$\max_{\{c_p,c_0,c_1,d,h\}} \{u(c_p) + \theta [u(c_0) + \beta u(c_1)]\}, \text{ subject to}$$
$$w_p + d = c_p + c_0 + h$$
$$c_1 = af(h) - Rd. \tag{1}$$

It is straightforward to show that unconstrained optimal investment is the same as in the paper (i.e. it solves $af'\left(h^U(a)\right) = R$). It is trivial to show that Lemma 1 holds (where $d$ now reflects total family borrowing/savings).

**Exogenous Credit Constraints** Now assume that the additional exogenous borrowing constraint is imposed: $d \leq d_0$.

If the constraint does not bind, investment is given by $h^U(a)$. However, if the borrowing constraint binds, the first order conditions become

$$u'(w_p + d_0 - c_0 - h) = \theta u'(c_0)$$
$$\beta af'(h)\, u'(af(h) - Rd_0) = u'(c_0).$$

To characterize $h^X(a, w_p)$ in this case, apply implicit differentiation and Cramer's rule:

$$\frac{\partial h^X(a,w_p)}{\partial a} = \frac{-[u''(c_p) + \theta u''(c_0)]}{|A|} \frac{\partial \mathbf{F}^X_{\mathbf{PAPER}}}{\partial a},$$

where $\mathbf{F}^X_{\mathbf{PAPER}}$ is the expression in our paper from which we derive our proposition for the exogenous constraints model. The matrix

$$A = \begin{bmatrix} u''(c_p) + \theta u''(c_0) & u''(c_p) \\ -u''(c_0) & \beta a \left\{ f''(h) u'[c_1] + a [f'(h)]^2 u''[c_1] \right\} \end{bmatrix}$$

has a positive determinant $|A| > 0$. Since $-[u''(c_p) + \theta u''(c_0)] > 0$,

$$sign \left\{ \frac{\partial h}{\partial a} \right\} = sign \left\{ \frac{\partial F^X_{\mathbf{PAPER}}}{\partial a} \right\}.$$

The main lesson is quite simple. Adding altruistic parents and intergenerational transfers does not affect the sign of the relationship between ability and investment in the young person's human capital. However, the magnitudes may change because the level of ability of the young individual will also impact the consumption of his parents.

For constrained families, one can similarly show that investment in the child's human capital will be strictly increasing in parental wealth, $w_p$. Thus, an extension of Proposition 1 of the paper holds referring to family wealth $w_p$ rather than the individual's wealth $w$.

**GSL** Now consider the case in which credit is available from GSL programs. Given the results in the previous section, it is easy to see that our characterization of the ability-investment relationship with altruistic parents is the same as in the paper. The GSL removes the conflict between net income maximization and consumption smoothing. The key assumption here is that lending is tied to the child's investment and cannot be used to finance the consumption of either the parent or the child. All qualitative results go through where wealth refers to $w_p$.

**Private lending with limited commitment** Consider now our benchmark $G+L$ model in the paper, and as in the paper, start with the case when $d_{\max} = \varrho(a) = 0$ and $\kappa > 0$. This is a special case of private lending only. In this environment, the family problem is subject to the endogenous debt constraint $d \leq \kappa a f(h)$.

Obviously, if the borrowing constraint does not bind, optimal investment is $h^L(a, w_p) = h^U(a)$. Now, assume that it binds. The first order conditions reduce to

$$u'(w + \kappa a f(h) - c_0 - h) = \theta u'(c_0),$$
$$\beta(1 - \kappa R) a f'(h) u'[(1 - \kappa R) a f(h)] = u'(c_0)[1 - \kappa a f'(h)].$$

Applying implicit differentiation and Cramer's rule,

$$\frac{\partial h^L(a, w_p)}{\partial a} = \frac{-[u''(c_p) + \theta u''(c_0)] \left( \frac{\partial \mathbf{F}^L_{\mathbf{PAPER}}}{\partial a} \right) + [-u''(c_0)[\kappa a f'(h) - 1]\{u''[c_p] \kappa f(h)\}]}{|A|} \tag{2}$$

where now

$$A=\left[\begin{array}{cc} u''(c_p)+\theta u''(c_0) & -u''[c_p][\kappa a f'(h)-1] \\ u''(c_0)[\kappa a f'(h)-1] & \frac{\partial \mathbf{F}^L_{\mathbf{PAPER}}}{\partial h} \end{array}\right] \tag{3}$$

and $\mathbf{F}^L_{\mathbf{PAPER}}$ is the expression in Appendix B of the paper that defines the first order conditions for the $G+L$ model when $d_{\max}=\varrho(a)=0$. It can be directly verified that $|A|>0$. In the numerator, the terms $-u''(c_0)[\kappa a f'(h)-1]\{u''[c_p]\kappa f(h)\}$ and $[-(u''(c_p)+\theta u''(c_0))]$ are both positive. Therefore, a sufficient condition for $\frac{\partial h^L(a,w_p)}{\partial a}>0$ is that the analogous expression in the model of our paper is positive (i.e. $\frac{\partial \mathbf{F}^L_{\mathbf{PAPER}}}{\partial a}>0$). It is straightforward to show that investment is strictly increasing in parental wealth, $w_p$, for constrained families.

**Baseline Model: Public and Private lending with Limited Commitment** From the GSL and private lending results just presented, it is evident that the analytical results for the baseline model in the paper can be extended to a model with endogenous parental transfers where both, private and public lending operate along the lines assumed in the paper. Define, as in the paper, the ability threshold $\bar{a}$ as the maximum ability for which the GSL credit suffices by itself to finance the unconstrained level of investment, i.e. $\bar{a}f'(d_{\max})=R$. Parallel arguments to those above indicate that Propositions 3 and 4 of the paper hold for $w_p$ instead of $w$. In particular, over-investment for $a<\bar{a}$ can arise as poor parents and children seek to expand their access to private credit as a means to increase their consumption.

In sum, we have shown that our results regarding the implied relationship between investment and ability are robust to the introduction of altruistic parents that endogenously determine $w$. This is important since "altruistic parents" is the leading economic model for parental transfers and bequests. All qualitative results with respect to initial wealth in the paper hold in this framework with respect to parental wealth, $w_p$.

We now briefly consider two other models of parental transfers popular in the literature.

## 2.2 Warm-Glow Preferences

A common (and simpler) alternative assumption in the literature is that parents do not value the utility of children but instead assign value to the amount of resources they transfer to them. In this model, the utility of parents is given by $U^P=u(c_p)+v(b)$, where $v$ is a strictly increasing and strictly concave function of bequests, $b$.

The problem of the parent is simply to maximize $U^P$, which leads to the bequest function $b^*(w_p)$. If Inada conditions for $u(\cdot)$ and $v(\cdot)$ hold, optimal bequests satisfy $u'(w_p-b)=v'(b)$, and bequests $b^*(w_p)$ are strictly increasing in $w_p$ and independent of ability.[1] Since

[1]If $v(\cdot)$ does not satisfy Inada conditions and youth have some initial wealth of their own (say from

$w_p$ determines $w$, the economic problems of the child (under all forms of constraints) are exactly the same as in the paper.

All results in our paper, without qualification, hold in this model, because bequests do not respond to ability $a$ or to the lending opportunities faced by the child. Since bequests are increasing in parental wealth, all of our qualitative results regarding the youth's initial wealth hold for both the child's wealth/bequests as well as parental wealth, $w_p$.

## 2.3 Paternalistic Preferences

Another commonly used intergenerational model assumes that parents take pride in the schooling attainment of their children. Parental preferences will be given $U^p = u(c_p) + v(h)$, where $v$ is an increasing and concave function. In this case, parental transfers depend positively on parental wealth/income $w_p$ and the child's schooling expenditures $h$. Keane and Wolpin (2001) essentially estimate a model consistent with this assumption.

A natural assumption in this model is that parents fully decide on the investments in their child. If so, human capital investments would be given by the condition $u'(w_p - h) = v'(h)$. In this case, human capital investments would be fully pinned down by the wealth of the family $w_p$ and *completely unrelated to the child's ability.* Given a positive intergenerational correlation in ability, the unconditional correlation between ability and investment should be positive. However, once we control for familial resources as in the paper, the correlation should be zero. This is clearly at odds with the evidence presented in our paper as well in a huge literature on schooling decisions. Moreover, the model would imply that investment should always be increasing in parental wealth, even for unconstrained individuals. This is also in stark contrast with the evidence. Therefore, even if the assumption of full parental control of $h$ is natural for pre-, primary, and secondary schooling, it is a bad assumption for college and beyond under paternalistic preferences.

More interestingly, let us assume that investment in human capital is the result of a simultaneous-moves game between the parent and the child. The parent influences the human capital decisions of the child by making transfers and the child influences the transfer decision-making of the parents by investing in schooling (i.e. what the parent cares about). The problem of parents is to allocate their initial wealth $w_p$ between their own consumption $c_p$ and transfers $\tau \geq 0$ to their child. The problem of children is to allocate their resources (foregone earnings $w_0^{fe}$ plus parental transfers $\tau$) between their own consumption $c_0$ and investment $h$. Given the parent's wealth $w_p$ and the child's ability $a$, the equilibrium in

foregone earnings), then bequests can be zero for poorer households and positive for richer households. In this case, $b^*(w_p)$ is still entirely pinned down by parental wealth $w_p$ and does not depend on the child's ability, investments, future consumption or borrowing.

this model is described by a pair of functions $\tau(h; w_P, a)$ and $h(\tau; w_P, a)$ that indicate, respectively, the best response of transfers to the investment of the child and the best response of investments to the transfers of the parent.

Our results continue to hold in this formulation under reasonable restrictions on the parental transfer decision. To this end, let $x$ denote the investments directly borne by the child. In terms of $x$, an equilibrium is a pair of functions $\{\tau(x), x(\tau)\}$ of best responses for parents and children, respectively. Parents make transfers $\tau = \tau(x)$, and children invest $x(\tau)$. Given $\tau(x)$, total human capital investments are $h = x + \tau(x)$. The period $t = 0$ budget constraint for the child is $c_0 + x \leq w_0^{fe} + d$, and his second period earnings are $y = af[x + \tau(x)] \equiv aH(x)$. The formulation of the game in terms of $x$ is equivalent to the formulation in terms of $h$ as long as $x > 0$.

As long as the resulting function $H(\cdot)$ is increasing and concave, this problem is the same as that of our paper where we now replace total investment $h$ with investment borne by the child $x$. As such, our comparative static results for the ability – investment relationship under exogenous borrowing constraints (EXC) and under private lending with limited commitment (LC) apply for $x$. If the GSL is defined such that youth cannot borrow more than they invest themselves (subject to upper limit $d_{max}$), those results go through as well. Moreover, as long as $h$ is increasing in $x$, our results on ability also hold for total investment, $h$. This will be the case as long as $\tau'(x) > -1$. In equilibrium, this condition should be satisfied locally, at least; otherwise, children would be strictly better off with lower $x$.

In sum, we show that simple and reasonable conditions on the behavior of transfers ensure that key results in the paper can be extended to parental wealth $w_p$.

# 3 Cost of investment dependent on ability

In the paper, we assume that the (marginal) cost of investment is invariant to the ability of the individual. Here, we examine whether our main results regarding the ability – investment relationship are robust to relaxing this assumption. A **decreasing cost** can arise, because more able students receive more fellowships and other forms of aid. In addition, more able students may find studying more enjoyable or easier. In these cases, the full cost (pecuniary and non-pecuniary) of investing $h$ is decreasing in ability $a$. By contrast, an **increasing cost** arises when time is an important input in human capital production and the value of time is increasing in ability $a$ (i.e. foregone earnings).

We first explore the implications of allowing the cost of investment to depend on ability. Then, we discuss the empirically relevant shape of the investment cost function, and whether our conclusions about the role of ability under exogenous constraints in the paper are warranted.

## 3.1 Implied behavior of investment

Assume that an individual with ability $a$ that invests $h$ units in human capital bears a total cost equal to $g(a)\,h$. The marginal (and average) cost of each unit invested is $g(a) > 0$, which may be increasing, decreasing, or constant in $a$.

### 3.1.1 Unrestricted investment

The unrestricted investment amount $h^U(a)$ is defined by $g(a) = R^{-1}af'(h)$, and

$$h^U(a) = f'^{-1}\left[\frac{Rg(a)}{a}\right]. \tag{4}$$

Unlike our baseline model, unconstrained investment can no longer be determined from the earnings function alone. It will also depend on the cost function $g(a)$, which may or may not be observable.

If we define $\varepsilon \equiv \frac{ag'(a)}{g(a)}$, the cost elasticity of investment, then it is natural to distinguish between a few different cases. If costs are very elastic to ability, $\varepsilon \geq 1$, then unconstrained investment is decreasing in ability, an empirically uninteresting case. Our baseline model implicitly assumes $\varepsilon = 0$. A modest positive elasticity, $0 < \varepsilon < 1$ yields a positive relationship between unconstrained investment and ability, but one which is weaker than our baseline model. A negative elasticity, $\varepsilon < 0$, yields a stronger ability – unconstrained investment relationship than our baseline case. Not surprisingly, the latter case will also tend to create a positive ability – investment relationship with exogenous constraints under more general conditions than our baseline model. In all other cases, we are more likely to observe a negative relationship between ability and investment than in our baseline model.

### 3.1.2 Exogenous borrowing constraints: $d \leq d_0$.

When the exogenous constraint binds, individuals

$$\max_h \left\{u\left[w + d_0 - g(a)\,h\right] + \beta u\left[af(h) - Rd_0\right]\right\}.$$

The first order condition is $F \equiv -u'(c_0)\,g(a) + \beta af'(h)\,u'(c_1) = 0$. Using implicit differentiation, we note that $sign\left\{\frac{\partial h^X}{\partial a}\right\} = sign\left\{\frac{\partial F}{\partial a}\right\}$. After simplifying:

$$\frac{\partial F}{\partial a} = \beta f'(h)\,u'(c_1)\left\{1 - \varepsilon + \frac{1}{\eta(c_0)}\left(\frac{\partial c_0}{\partial a}\frac{a}{c_0}\right) - \frac{1}{\eta(c_1)}\left(\frac{\partial c_1}{\partial a}\frac{a}{c_1}\right)\right\}, \tag{5}$$

where $\eta(c_i) \equiv -u'(c_i)\,/\,[c_i u''(c_i)]$ is the IES and $\varepsilon$ is the cost elasticity of investment as defined above.

In the baseline model, $\varepsilon = 0$ and $\frac{\partial c_0}{\partial a} = 0$. Therefore, $\frac{\partial F}{\partial a} = \beta f'(h) u'(c_1) \left\{ 1 - \frac{1}{\eta(c_1)} \left( \frac{\partial c_1}{\partial a} \frac{a}{c_1} \right) \right\}$. Since $\frac{\partial c_1}{\partial a} \frac{a}{c_1} = \frac{af(h)}{af(h) - Rd_0} \geq 1$, a sufficient condition for $\frac{\partial F}{\partial a} < 0$ is $\eta(c_1) < 1$, as stated in the paper.

For the general case, observe that $\frac{\partial c_0}{\partial a} \frac{a}{c_0} = \frac{-g'(a)h}{w + d_0 - g(a)h} a = -\varepsilon \left( \frac{g(a)h}{w + d_0 - g(a)h} \right)$. If $\varepsilon > 0$ (i.e. cost of investment is increasing in ability as with foregone earnings) then $\frac{\partial c_0}{\partial a} \frac{a}{c_0} < 0$ and the perverse ability – investment relationship of Proposition 1 holds more strongly than in the paper. That is, investment is decreasing in ability for an even larger range of IES values than our baseline model with exogenous constraints.[2] The more the investment costs increase in ability, the higher the $IES$ must be for the exogenous constraint model to predict a positive relationship.

Unfortunately, no general analytical results for $\frac{\partial h^X}{\partial a}$ can be obtained when $\varepsilon < 0$. As such, we numerically explore the implied ability – investment relationship in this case by extending the *quantitative* model of the paper to allow for general costs of investment, $g(a) = (a/a_{\text{low}})^{\varepsilon}$ for a fixed value $\varepsilon$. We take all other parameter values from the baseline calibration. Here $a_{\text{low}}$ is an arbitrary lower bound on which the marginal cost of investment is normalized to one. Our baseline model in the paper is $\varepsilon = 0$ because $g(a) = 1$ for all ability levels $a$.

Figure 1 shows the behavior of constrained investment for our baseline case $\varepsilon = 0$, two cases in which investment costs decrease with ability ($\varepsilon \in \{-0.5, -0.25\}$), and two cases in which investment costs increase with ability ($\varepsilon \in \{0.25, 0.5\}$).[3] As already discussed, for values of $\varepsilon \leq 0$, we obtain a negative ability – investment relationship (when the constraint binds) given our IES of 0.5. More interestingly, we also obtain a strong negative relationship for $\varepsilon = -0.25$. Only when the cost elasticity of investment to ability falls to $\varepsilon = -0.5$ do we obtain a roughly flat (but still negative) ability – investment relationship. In sum, investment costs must be very strongly declining in ability to overcome the desire that constrained individuals have to smooth consumption.

Of course, such a strong negative elasticity of investment costs to ability also has significant effects on the relationship between ability and *unconstrained* investment levels. Indeed, $\varepsilon = -0.5$ implies that *unconstrained* investment for the highest AFQT quartile should be 23 times the investment of the lowest AFQT quartile. This ratio is unrealistically high. The observed ratio in the NLSY79 is much lower, only 12.4, and much closer to the implied ratio of 11.5 in our calibrated model.

[2] If $\varepsilon \geq 1$, then $\frac{\partial h^X}{\partial a} < 0$ for any IES, while for $0 < \varepsilon < 1$ a sufficient condition for $\frac{\partial h^X}{\partial a} < 0$ is $\eta(c_1) \leq \frac{1}{1-\varepsilon}$.

[3] For illustration purposes, we have extended the range of abilities considered so that $a_{\text{low}}$ is half the estimated value for the lowest AFQT quartile (i.e. $a_{\text{low}} = 1/2 \times a_1$). The maximum is twice the estimated value for the highest quartile (i.e. $2 \times a_4$).

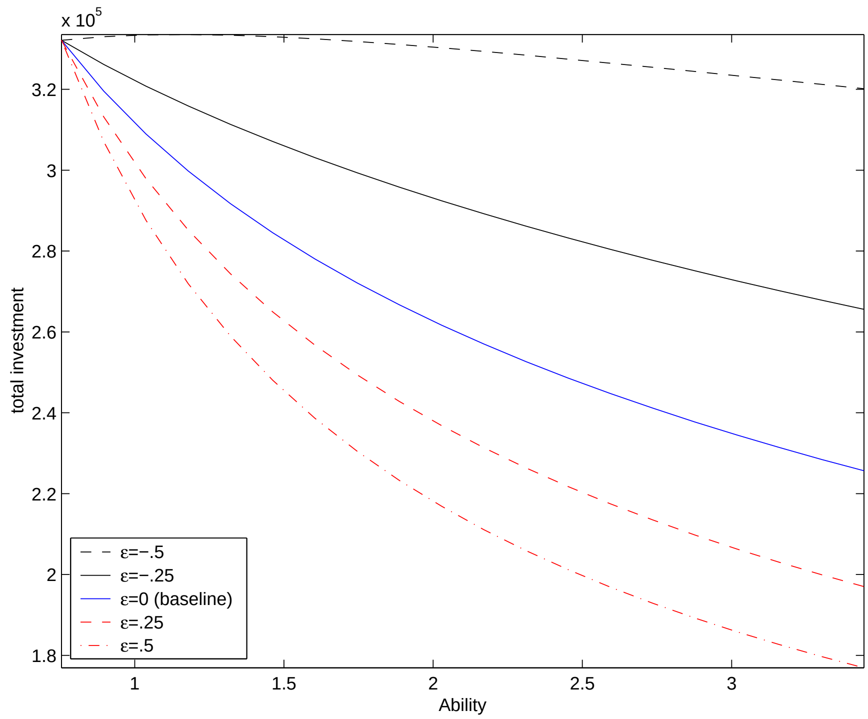


Figure 1: Exogenous Contraints Model: Implied Contrained Investment-Ability relationship for the model extend with $g(a) = \varepsilon_0 a^{\varepsilon}$ for $\varepsilon \in \{-.5, .25, 0, .25, .5\}$

## 3.2 Discussion

The baseline model in the paper assumes investment costs are independent of ability. To the extent that foregone earnings are increasing in ability, this suggests investment costs should be increasing in ability. We show that this only strengthens our finding that an exogenous constraint model predicts a negative ability – investment relationship among constrained youth.[4]

Alternatively, if the marginal cost of ability is strongly decreasing in ability, either because smarter individuals find it easier to study or because they receive merit-based aid, it is theoretically possible to obtain a positive ability – investment relationship among constrained youth. Empirically, however, this seems extremely unlikely.

First, there is little evidence to suggest that the direct costs of investment are systematically decreasing in ability. If anything, merit aid is only likely to be important for the very high end of the ability distribution, while it is very low throughout the rest of the distribution. In the 1999-2000 academic year, fewer than 10% of all students at public post-secondary institutions received non-need based grants from the state or their institutions (Heller, 2003).

[4] For similar reasons, this would also tend to weaken any positive effects of ability on investment when individuals are constrained by private lenders as modeled in the paper.

Comparing this against our estimated 50 percentage point gaps in college attendance rates between the highest and lowest ability quartiles, it is clear that merit aid explains very little, if any, of the observed positive ability – schooling relationship. Although, merit aid is only relevant at the top end of the ability distribution, we observe a strong positive relationship between ability and schooling even at the bottom end. In sum, the data largely supports our assumption of a uniform net tuition cost of investment, at least for all except the very brightest.

Second, our calibrated model (with marginal costs the same for all abilities) is able to simultaneously explain both the relationship between ability, schooling and earnings as well as the relationship between schooling and ability. If more able people enjoyed school much more than less able people, there should be much larger gaps in schooling by ability than our model can account for through earnings differences alone. More specifically, we show that the cost elasticity with respect to ability would need to be very high to generate a non-negative ability – investment relationship in the exogenous constraint model. In fact, this elasticity would have to be so high that it implies an implausibly strong ability – investment relationship for unconstrained youth relative to that observed in the NLSY data.

Finally, we note that pure non-pecuniary factors (i.e. schooling in the utility function) can produce odd predictions about measured returns to human capital investment. If all individuals are unconstrained (consistent with the NLSY79) and more able people enjoy school more, then measured marginal returns to human capital investment should be declining in ability. Indeed, when we take $\varepsilon = -0.5$ in our calibrated model, the returns to investment for the highest AFQT quartile are only 90% of the returns for the least able.

# 4 Generalizing the human capital production function

To explore the importance of our assumptions about the human capital production function, we can generalize post-school earnings (in the two-period model) as follows: let $y = f(a, h)$, where $\frac{\partial f}{\partial h} > 0$, $\frac{\partial^2 f}{\partial h^2} < 0$, $\frac{\partial f}{\partial a} > 0$, and $\frac{\partial^2 f}{\partial h \partial a} > 0$. Otherwise, consider the same problem as in the paper.

Without borrowing constraints, optimal investment solves $\frac{\partial f}{\partial h} = R$ and optimal unconstrained investment, $h^U(a)$, is an increasing function of ability and is independent of $w$. As the following analysis shows, the elasticity of substitution between investment and ability in the production of human capital and the consumption intertemporal elasticity of substitution (IES) play key roles in determining the relationship between ability and investment for constrained borrowers. With preferences defined by a constant intertemporal elasticity of substitution (elasticity $\eta$) and a CES human capital production function (elasticity of substitution, $\phi$), we obtain a negative ability investment relationship for youth constrained by exogenous borrowing constraints if $\eta < \phi$. In the case of endogenous constraints generated by limited commitment, we obtain a positive ability – investment relationship if $\eta > (1 - R\kappa)\phi$, so there are parameterizations that imply a negative ability – investment relationship under exogenous constraints but not under endogenous constraints. Under the GSL, investment behaves largely as discussed in the text. The discussion below provides a more detailed characterization of investment behavior assuming a general human capital production function.

When ability and investment are strong complements (i.e. $\phi < 1$), it may be possible to obtain a positive ability – investment relationship under exogenous constraints when the IES $\eta < 1$. The vast majority of empirical and theoretical studies on schooling, ability, and earnings assume a multiplicatively separable relationship between schooling and ability. This assumption fits our data quite well, and we are not aware of any compelling evidence that suggests serious departures from this assumption. As such, the paper assumes multiplicative separability, which is equivalent to the more general case presented here with $\phi = 1$.

## 4.1 Exogenous borrowing constraints

Now, consider the exogenous borrowing constraint $d \leq d_0$. Imposing the borrowing constraint, $d = d_0$, constrained borrowers $\max_h \; u(w + d_0 - h) + \beta u(f(a,h) - Rd_0)$.

This yields FOC for investment:

$$-u'(w + d_0 - h) + \beta u'(f(a,h) - Rd_0)\frac{\partial f}{\partial h} = 0,$$

which implicitly defines optimal investment for constrained borrowers, $h^X(a,w)$, as a function of ability and initial assets. It is straightforward to show that investment is increasing in initial wealth, $w$.

Using implicit differentiation, one can show that for constrained persons

$$\begin{aligned} sign\left\{\frac{dh^X}{da}\right\} &= sign\left\{u''(c_2)\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right) + u'(c_2)\frac{\partial^2 f}{\partial h \partial a}\right\} \\ &= sign\left\{\eta(c_2) - \left[\frac{\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right)}{f(a,h)\left(\frac{\partial^2 f}{\partial h \partial a}\right)}\right]\left(\frac{f(a,h)}{f(a,h) - Rd_0}\right)\right\}. \end{aligned} \tag{6}$$

where $\eta(c_2) = \frac{-u'(c_2)}{c_2 u''(c_2)}$ is the consumption intertemporal elasticity of substitution (at $c_2$).

Thus,

$$\frac{dh^X}{da} < 0 \quad \Leftrightarrow \quad \eta(c_2) < \left[\frac{\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right)}{f(a,h)\left(\frac{\partial^2 f}{\partial h \partial a}\right)}\right]\left(\frac{f(a,h)}{f(a,h) - Rd_0}\right).$$

For $d_0 \geq 0$, $f(a,h) \geq f(a,h) - Rd_0$ and

$$\frac{dh^X}{da} < 0 \quad if \quad \eta(c_2) < \frac{\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right)}{\left(\frac{\partial^2 f}{\partial h \partial a}\right) f(a,h)}. \tag{7}$$

If $f(a,h)$ is of CES form with elasticity of substitution $\phi$, then

$$\frac{dh^X}{da} < 0 \quad if \quad \eta(c_2) < \phi.$$

The case in our paper is equivalent to this specification with $\phi = 1$.

## 4.2 GSL system

Investment behaves as in the paper (where the new $h^X(a,w)$ replaces that of the text).

## 4.3 Private lenders with limited commitment

Now, consider our private lending constraint: $d \leq \kappa f(a,h)$. Those constrained by this endogenous borrowing limit solve the following maximization problem:

$$\max_h \{u(w + \kappa f(a,h) - h) + \beta u[(1 - R\kappa) f(a,h)]\}.$$

This problem yields the FOC for investment:

$$F(h,a) \equiv u'(c_1)\left[\kappa\left(\frac{\partial f}{\partial h}\right) - 1\right] + \beta u'(c_2)(1 - R\kappa)\left(\frac{\partial f}{\partial h}\right) = 0,$$

which implicitly defines optimal investment, $h^L(a,a)$, for constrained borrowers.

One can easily show that investment is increasing in initial assets, $a$. Regarding ability, one can show that $sign\{\frac{dh^L}{da}\} = sign\{\frac{\partial F}{\partial a}\}$. Note

$$\begin{aligned}\frac{\partial F}{\partial a} &= u''(c_1)\kappa\left(\frac{\partial f}{\partial a}\right)\left[\kappa\left(\frac{\partial f}{\partial h}\right) - 1\right] + u\prime(c_1)\kappa\left(\frac{\partial^2 f}{\partial h \partial a}\right) \\ &+ \beta\left[u''(c_2)(1 - R\kappa)^2\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right) + u'(c_2)(1 - R\kappa)\left(\frac{\partial^2 f}{\partial h \partial a}\right)\right].\end{aligned}$$

Dividing this through by $-\beta(1 - R\kappa)c_2 u''(c_2)\frac{\partial^2 f}{\partial h \partial a}$ and re-arranging terms:

$$\begin{aligned}&sign\left\{\frac{dh^L}{da}\right\} \\ &= sign\left\{\eta(c_2) - \left[\frac{\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right)}{f(a,h)\left(\frac{\partial^2 f}{\partial h \partial a}\right)}\right] + \frac{u''(c_1)\kappa\left(\frac{\partial f}{\partial a}\right)\left[1 - \kappa\left(\frac{\partial f}{\partial h}\right)\right]}{\beta(1 - R\kappa)c_2 u''(c_2)\left(\frac{\partial^2 f}{\partial h \partial a}\right)} + \frac{-u'(c_1)\kappa}{\beta(1 - R\kappa)c_2 u''(c_2)}\right\},\end{aligned}$$

where we use the definition of $\eta(c_2)$ and $c_2 = (1 - R\kappa)f(a,h)$. Because $u'(c_1) > \beta R u'(c_2)$ when the borrowing constraint binds, the third term is positive and

$$sign\left\{\frac{dh^L}{da}\right\} > 0 \quad \text{if} \quad \eta(c_2) > (1 - R\kappa)\left[\frac{\left(\frac{\partial f}{\partial h}\right)\left(\frac{\partial f}{\partial a}\right)}{f(a,h)\left(\frac{\partial^2 f}{\partial h \partial a}\right)}\right].$$

Clearly, there are parameterizations where $\frac{dh^L}{da} > 0$ but $\frac{dh^X}{da} < 0$, so this model model can produce a positive relationship between investment and ability in cases where the exogenous constraint model does not.

If $f(a,h)$ is CES with elasticity of substitution $\phi$,

$$sign\left\{\frac{dh^L}{da}\right\} > 0 \quad \text{if} \quad \eta(c_2) > (1 - R\kappa)\phi.$$

The case in the paper is equivalent to $\phi = 1$.