Online Appendix to "Communication in Cournot Oligopoly"

Maria Goltsman* Gregory Pavlov†

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This appendix contains the proofs of Lemmas 1 and 6-9, as well as supplementary lemmas A.1 and A.2 and calculations for Examples 1 and 4.

1 Proofs of Section 3

Lemma A.1

(i) $\rho(q_i) - \beta q_{-i} \geq 0$ for every pair (q_i, q_{-i}) that is rationalizable for some for some (c_i, c_{-i}).

(ii) Suppose $C(q_i, c_i)$ is C^2 in q_i, $\frac{\partial C_i(q_i,c_i)}{\partial q_i}$ is C^1 in c_i, ρ is C^2, and, for some $\varepsilon > 0$, $\rho''(q_i)q_i + (1-\varepsilon)\rho'(q_i) < 0$ for every q_i. Then $q(q_{-i}, c_i)$ is single-valued, continuous at every (q_{-i}, c_i), C^1 on $\{(q_{-i}, c_i) : q(q_{-i}, c_i) > 0\}$. If $q(q_{-i}, c_i) > 0$, then $\frac{\partial q(q_{-i},c_i)}{\partial c_i} \leq 0$ and $\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}} \in (-\frac{1}{1+\varepsilon}, 0)$.

(iii) Suppose C1 and C2 hold, and $\frac{\partial C(0,c_i)}{\partial q_i} = 0$ for every $c_i \in C$. Then $q(q_{-i}, c_i) > 0$ for every $q_{-i} \in [0, q(0,0)]$ and every $c_i \in C$.

*Department of Economics, University of Western Ontario, Canada

†Corresponding author. Department of Economics, University of Western Ontario, Canada

Proof. (i) Let $\overline{q}$ be the revenue-maximizing output when $q_{-i} = 0$, i.e. $\overline{q} = \arg\max_{q_i \geq 0} P(q_i, 0)\, q_i$. Since $|\rho'(q_i)| \geq \beta$, $\overline{q}$ cannot be greater than $\frac{\rho(0)}{\beta}$. This, together with the fact that the revenue is continuous in q_i, implies that $\overline{q}$ exists. Since the revenue is zero at $q_i = 0$ and $q_i = \frac{\rho(0)}{\beta}$, the solution is interior and satisfies the first-order condition: $\rho'(\overline{q})\, \overline{q} + \rho(\overline{q}) = 0$.

Note that no type $c_i \in C$ will find it optimal to choose output higher than $\overline{q}$ regardless of the conjecture about the opponent's play. This is because such outputs result in (weakly) lower revenue than $\overline{q}$ (not just when $q_{-i} = 0$, but for every $q_{-i} \geq 0$), and strictly higher cost (because $\frac{\partial C(q_i, c_i)}{\partial q_i} > 0$ when $q_i > 0$). Hence, if (q_i, q_{-i}) is rationalizable, then

$$\rho(q_i) - \beta q_{-i} \geq \rho(\overline{q}) - \beta \overline{q} = (-\rho'(\overline{q}) - \beta)\, \overline{q} \geq 0$$

where the first inequality is because $\rho' < 0$ and $\beta > 0$, the equality is by definition of $\overline{q}$, and the second inequality is due to $|\rho'(q)| \geq \beta$.

(ii) Note that

$$\frac{\partial^2 \pi_i(q_i, q_{-i}, c_i)}{\partial q_i^2} = \rho''(q_i) q_i + 2\rho'(q_i) - \frac{\partial^2 C(q_i, c_i)}{\partial q_i^2} < (1+\varepsilon)\, \rho'(q_i) \leq -(1+\varepsilon)\, \beta < 0 \quad (1)$$

for every $q_i \geq 0$. Thus π_i is strictly concave in q_i, and therefore q is single-valued. By the Theorem of the Maximum, q is continuous in (q_{-i}, c_i). Note that q equals 0 if $\rho(0) - \beta q_{-i} - \frac{\partial C_i(0, c_i)}{\partial q_i} \leq 0$, and solves the first-order condition

$$\rho'(q_i) q_i + \rho(q_i) - \beta q_{-i} - \frac{\partial C_i(q_i, c_i)}{\partial q_i} = 0$$

otherwise. By the Implicit Function Theorem, q is continuously differentiable in (q_{-i}, c_i)

whenever $q(q_{-i}, c_i) > 0$, i.e. $\rho(0) - \beta q_{-i} - \frac{\partial C_i(0,c_i)}{\partial q_i} > 0$, with

$$\frac{\partial q(q_{-i}, c_i)}{\partial c_i} = \frac{\frac{\partial^2 C_i(q_i,c_i)}{\partial q_i \partial c_i}}{\frac{\partial^2 \pi_i(q_i,q_{-i},c_i)}{\partial q_i^2}} \leq 0, \; \frac{\partial q(q_{-i}, c_i)}{\partial q_{-i}} = \frac{\beta}{\frac{\partial^2 \pi_i(q_i,q_{-i},c_i)}{\partial q_i^2}}.$$

Using (1) we get $\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}} \in \left(-\frac{1}{1+\varepsilon}, 0\right)$.

(iii) Let $\overline{q}$ be as defined in part (i). Then

$$\begin{aligned}
\frac{\partial \pi(0, q_{-i}, c_i)}{\partial q_i} &= \rho(0) - \beta q_{-i} - \frac{\partial C(0, c_i)}{\partial q_i} \\
&\geq \rho(0) - (-\rho'(\overline{q}))\, \overline{q} \geq \rho(0) - \rho(\overline{q}) > 0
\end{aligned}$$

where the first inequality uses the facts that that $\beta \leq -\rho'(\overline{q})$, $q_{-i} \leq \overline{q}$, and $\frac{\partial C(0,c_i)}{\partial q_i} = 0$; the second inequality uses the first-order condition for $\overline{q}$. Thus $q(q_{-i}, c_i) > 0$ for every $q_{-i} \in [0, q(0,0)] \subseteq [0, \overline{q}]$. ■

2 Proofs of Section 4

Proof of Lemma 1. Let

$$BR_i(q_{-i}) = \int q(q_{-i}, c_i)\, dF_i(c_i) \qquad \text{for } i \in \{A, B\}$$

The expected outputs in a Bayesian-Nash equilibrium satisfy

$$Q_A = BR_A(Q_B),\; Q_B = BR_B(Q_A) \tag{2}$$

Let $H(q_A, q_B) = (BR_A(q_B), BR_B(q_A))$. By C2, H maps the interval $[0, q(0,0)]^2$ into itself. C2 also implies that for every c_i and $Q_{-i} \neq Q'_{-i}$:

$$\left|q(Q_{-i}, c_i) - q\left(Q'_{-i}, c_i\right)\right| < (1-\delta)\left|Q_{-i} - Q'_{-i}\right| \tag{3}$$

This in turn implies that H is a contraction mapping in the sup norm.

Consider the sequence $\left\{Q_A^k, Q_B^k\right\}_{k=0}^{\infty}$ defined by

$$Q_A^0 = Q_B^0 = 0;$$
$$(Q_A^k, Q_B^k) = H(Q_A^{k-1}, Q_B^{k-1}),\ \ k \geq 1$$

and for $k \geq 1$, let

$$I_i^k = \left[\min\left\{Q_i^{k-1}, Q_i^k\right\}, \max\left\{Q_i^{k-1}, Q_i^k\right\}\right]$$

Because H is a contraction mapping on $[0, q(0,0)]^2$, the sequence $\left\{Q_A^k, Q_B^k\right\}_{k=0}^{\infty}$ converges. By continuity of BR_i, its limit satisfies (2) and thus defines the expected outputs in a Bayesian-Nash equilibrium.

Next, let us prove that any strategy $q_i(c_i)$ of firm i that survives k rounds of elimination of interim strictly dominated strategies has to satisfy $\int q_i(c_i)dF_i(c_i) \in I_i^k$. Indeed, the statement holds for $k = 1$: for every i, $\int q_{-i}(c_{-i})dF_{-i}(c_{-i}) \geq 0$ implies that any strategy $q_i(c_i)$ such that $q_i(c_i) > q(0, c_i)$ is interim strictly dominated for type c_i. Thus the first round of elimination leaves only strategies such that $\int q_i(c_i)dF_i(c_i) \in [0, BR_i(0)] = I_i^1$. Suppose that the statement holds for $k \geq 1$, i.e. k rounds of elimination result in strategies for firm $-i$ such that $\int q_{-i}(c_{-i})dF_{-i}(c_{-i}) \in I_{-i}^k$. Conditional on firm $-i$ using such strategies, any strategy $q_i(c_i)$ of firm i such that

$$\begin{aligned} q_i(c_i) &\notin \left[q(\max\left\{Q_{-i}^{k-1}, Q_{-i}^k\right\}, c_i), q(\min\left\{Q_{-i}^{k-1}, Q_{-i}^k\right\}, c_i)\right] \\ &= \left[\min\left\{q(Q_{-i}^{k-1}, c_i), q(Q_{-i}^k, c_i)\right\}, \max\left\{q(Q_{-i}^{k-1}, c_i), q(Q_{-i}^k, c_i)\right\}\right] \end{aligned}$$

is interim strictly dominated for type c_i. Therefore, firm i's strategies surviving $k+1$

rounds of elimination satisfy

$$\int q_i(c_i)dF_i(c_i) \in \left[\min\left\{BR_i(Q_{-i}^{k-1}), BR_i(Q_{-i}^{k})\right\}, \max\left\{BR_i(Q_{-i}^{k-1}), BR_i(Q_{-i}^{k})\right\}\right]$$
$$= \left[\min\left\{Q_i^k, Q_i^{k+1}\right\}, \max\left\{Q_i^k, Q_i^{k+1}\right\}\right] = I_i^{k+1}$$

Let $(Q_A, Q_B) = \lim_{k\to\infty}(Q_A^k, Q_B^k)$ be the equilibrium expected outputs. Then

$$Q_i = \lim_{k\to\infty} \min\left\{Q_i^{k-1}, Q_i^k\right\} = \lim_{k\to\infty} \max\left\{Q_i^{k-1}, Q_i^k\right\} \text{ for } i = A, B.$$

Therefore, any strategy profile that survives iterated elimination of interim strictly dominated strategies has to satisfy $\int q_i(c_i)dF_i(c_i) = Q_i$, and the only strategy profile that survives the elimination is the one satisfying $q_i(c_i) = q(Q_{-i}, c_i)$, which is the condition for the Bayesian-Nash equilibrium. ■

Calculations for Example 4. Consider firm i with cost type c_i facing the opponent whose output is distributed with mean μ_{-i} and variance σ_{-i}^2. The expected profit of this firm is

$$\left(40 - q_i - \frac{1}{10}\mu_{-i} - \frac{1}{1000}\mu_{-i}q_i^2 - \frac{1}{1000}\left(\mu_{-i}^2 + \sigma_{-i}^2\right)q_i\right)q_i - c_i q_i$$

and the optimal output $q_i\left(\mu_{-i}, \sigma_{-i}^2, c_i\right)$ equals

$$\frac{\sqrt{\left(1 + \frac{1}{1000}\left(\mu_{-i}^2 + \sigma_{-i}^2\right)\right)^2 + \frac{3}{1000}\mu_{-i}\left(40 - \frac{1}{10}\mu_{-i} - c_i\right)} - \left(1 + \frac{1}{1000}\left(\mu_{-i}^2 + \sigma_{-i}^2\right)\right)}{\frac{3}{1000}\mu_{-i}}$$

if $40 - \frac{1}{10}\mu_{-i} - c_i \geq 0$, and 0 otherwise. It is straightforward to check that q_i is continuous, and, whenever $q_i > 0$, q_i is C^1, q_i is decreasing in μ_{-i}, σ_{-i}^2, and c_i, and $\left|\frac{\partial q_i}{\partial \mu_{-i}}\right| < 1$.

If we consider the maximized profit Π_i as a function of $\left(\mu_{-i}, \sigma_{-i}^2, c_i\right)$, then by the

Envelope theorem

$$\frac{\frac{d\Pi_i}{d\mu_{-i}}}{\frac{d\Pi_i}{d\sigma_{-i}^2}} = \frac{\left(-\frac{1}{10} - \frac{1}{1000}q_i^2 - \frac{1}{500}\mu_{-i}q_i\right) q_i}{-\frac{1}{1000}q_i^2} = \frac{100}{q_i} + q_i + 2\mu_{-i}$$

Thus the rate at which firm i is willing to substitute μ_{-i} for σ_{-i}^2 is nonmonotonic in q_i, and, since optimal q_i decreases in c_i, this rate is nonmonotonic in c_i.

If we take $c_L = 0$, $c_M = 12$, $c_H = 25.1167$, then there exists an informative cheap talk equilibrium where type c_M of firm A sends message m, while types c_L and c_H send m'; and firm B plays a babbling strategy. The approximate equilibrium outputs, the averages and the variances of outputs (computed numerically) are given below.

	after message m	after message m'
$q_A\left(\mu_B, \sigma_B^2, c_L\right)$	-	13.55931
$q_A\left(\mu_B, \sigma_B^2, c_M\right)$	9.84820	-
$q_A\left(\mu_B, \sigma_B^2, c_H\right)$	-	5.35421
μ_A	9.84820	9.45676
σ_A^2	0	16.83093
$q_B\left(\mu_A, \sigma_A^2, c_L\right)$	14.82376	14.83104
$q_B\left(\mu_A, \sigma_A^2, c_M\right)$	10.75555	10.74687
μ_B	12.78965	12.78895
σ_B^2	4.13757	4.17011

The approximate profits are as follows:

	after message m	after message m'
$\Pi_A\left(\mu_B,\sigma_B^2,c_L\right)$	278.45689	278.45690
$\Pi_A\left(\mu_B,\sigma_B^2,c_M\right)$	137.68507	137.68501
$\Pi_A\left(\mu_B,\sigma_B^2,c_H\right)$	37.40187	37.40193
$\Pi_B\left(\mu_A,\sigma_A^2,c_L\right)$	305.21566	305.03313
$\Pi_B\left(\mu_A,\sigma_A^2,c_M\right)$	151.40827	151.24357

The profit of type c_M of firm A is higher after message m than after m', and thus it prefers to send m. The profit of types c_L and c_H is higher after message m', so they prefer to send m'.

3 Proofs of Section 5

Lemma A.2 Let $r\left(q_i\right)=\rho'\left(q_i\right)q_i+\rho\left(q_i\right)$, and suppose that it is non-increasing. Denote the elasticities of $r_q\left(q_i\right)$, $C_{qq}\left(q_i,c_i\right)$, and $C_{qc}\left(q_i,c_i\right)$ by ε_{r_q}, $\varepsilon_{C_{qq}}$, and $\varepsilon_{C_{qc}}$. Then, for every (q_{-i},c_i) such that $q(q_{-i},c_i)>0$,

$$\frac{\partial^2\ln\left(q\left(q_{-i},c_i\right)\right)}{\partial c_i\partial q_{-i}}=\left(\frac{\beta C_{qc}}{q^2}\right)\frac{\left(\varepsilon_{C_{qc}}-\varepsilon_{r_q}-1\right)\left(-r_q\right)+\left(\varepsilon_{C_{qc}}-\varepsilon_{C_{qq}}-1\right)C_{qq}}{\left(-r_q+C_{qq}\right)^3} \tag{4}$$

Since $\beta,q>0$ and $-r_q,C_{qc},C_{qq}\geq 0$, (4) is more likely to be negative the lower is $\varepsilon_{C_{qc}}$ and the higher are ε_{r_q} and $\varepsilon_{C_{qq}}$.

Proof. From the first-order condition we can find

$$\frac{\partial q\left(q_{-i},c_i\right)}{\partial c_i}=-\frac{C_{qc}}{-r_q+C_{qq}},\ \frac{\partial q\left(q_{-i},c_i\right)}{\partial q_{-i}}=-\frac{\beta}{-r_q+C_{qq}} \tag{5}$$

and

$$\frac{\partial^2 q\left(q_{-i},c_i\right)}{\partial q_{-i}\partial c_i}=\frac{\beta\left(\left(-r_{qq}+C_{qqq}\right)\frac{\partial q}{\partial c_i}+C_{qqc}\right)}{\left(-r_q+C_{qq}\right)^2}=\left(\frac{\beta C_{qc}}{q}\right)\frac{\left(\varepsilon_{C_{qc}}-\varepsilon_{r_q}\right)\left(-r_q\right)+\left(\varepsilon_{C_{qc}}-\varepsilon_{C_{qq}}\right)C_{qq}}{\left(-r_q+C_{qq}\right)^3} \tag{6}$$

Note that

$$\frac{\partial^2 \ln\left(q\left(q_{-i},c_i\right)\right)}{\partial c_i\partial q_{-i}}=\frac{d}{dc_i}\left(\frac{\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}}}{q\left(q_{-i},c_i\right)}\right)=\frac{\frac{\partial^2 q(q_{-i},c_i)}{\partial q_{-i}\partial c_i}q\left(q_{-i},c_i\right)-\frac{\partial q(q_{-i},c_i)}{\partial q_{-i}}\frac{\partial q(q_{-i},c_i)}{\partial c_i}}{\left(q\left(q_{-i},c_i\right)\right)^2} \tag{7}$$

Substituting (5) and (6) in (7) yields the result. ■

Proof of Lemma 6. Let

$$\Phi\left(Q_{-i},c^*\right)=Q_{-i}-\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty}q\left(Q_{-i},c_i\right)dF\left(c_i\right)$$

Then $\Phi\left(Q^{H2}(c^*),c^*\right)=0$.

Note that Φ is continuous in all variables by C1 and the continuity of F;

$$\Phi\left(0,c^*\right)=-\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty}q\left(0,c_i\right)dF\left(c_i\right)<0$$

by C3. Let $Q'_{-i}>Q_{-i}$; then

$$\Phi(Q'_{-i},c_i)-\Phi(Q_{-i},c_i)=Q'_{-i}-Q_{-i}-\frac{1}{1-F\left(c^*\right)}\int_{c^*}^{\infty}\left(q\left(Q'_{-i},c_i\right)-q\left(Q_{-i},c_i\right)\right)dF\left(c_i\right)$$

$$\geq Q'_{-i}-Q_{-i}$$

where the inequality is by C2. Therefore there is a unique value of Q_{-i} such that $\Phi(Q_{-i},c^*)=0$, which we will call $Q^{H2}\left(c^*\right)$. The function $Q^{H2}\left(c^*\right)$ is continuous by Theorem 2.1 in Jittorntrum (1978). Let us prove that $Q^{H2}\left(c^*\right)$ is decreasing in c^*. If

$\tilde{c}^* < c^*$, and $Q^{H2}(\tilde{c}^*) < Q^{H2}(c^*)$, then

$$\begin{aligned}
&Q^{H2}(c^*) - Q^{H2}(\tilde{c}^*) \\
&= \frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q\left(Q^{H2}(c^*), c_i\right) dF(c_i) - \frac{1}{1-F(\tilde{c}^*)}\int_{\tilde{c}^*}^{\infty} q\left(Q^{H2}(\tilde{c}^*), c_i\right) dF(c_i) \\
&\leq \frac{1}{1-F(c^*)}\int_{c^*}^{\infty} q\left(Q^{H2}(\tilde{c}^*), c_i\right) dF(c_i) - \frac{1}{1-F(\tilde{c}^*)}\int_{\tilde{c}^*}^{\infty} q\left(Q^{H2}(\tilde{c}^*), c_i\right) dF(c_i) \\
&= \frac{F(c^*) - F(\tilde{c}^*)}{(1-F(\tilde{c}^*))(1-F(c^*))}\int_{c^*}^{\infty} q\left(Q^{H2}(\tilde{c}^*), c_i\right) dF(c_i) - \frac{1}{1-F(\tilde{c}^*)}\int_{\tilde{c}^*}^{c^*} q\left(Q^{H2}(\tilde{c}^*), c_i\right) dF(c_i) \\
&\leq \frac{F(c^*) - F(\tilde{c}^*)}{1-F(\tilde{c}^*)} q\left(Q^{H2}(\tilde{c}^*), c^*\right) - \frac{F(c^*) - F(\tilde{c}^*)}{1-F(\tilde{c}^*)} q\left(Q^{H2}(\tilde{c}^*), c^*\right) = 0
\end{aligned}$$

where both inequalities follow from C2. By definition,

$$Q^{H2}(0) = \int_0^{\infty} q\left(Q^{H2}(0), c_i\right) dF(c_i)$$

and therefore $Q^{H2}(0) = Q^{NC}$. Finally, $\lim_{c^* \to \infty} Q^{H2}(c^*) = 0$ by C6. ■

Proof of Lemma 7. Denote

$$\overline{\Psi}\left(Q_{-i}^L, c^*\right) = Q_{-i}^L - \int_0^{c^*} q\left(Q_{-i}^L, c_i\right) dF(c_i) - \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q\left(Q_{-i}^L, \widehat{c}\right) dF(\widehat{c}), c_i\right) dF(c_i)$$

Note that $Q_{-i}^L(c^*)$ is defined by $\overline{\Psi}\left(Q_{-i}^L(c^*), c^*\right) = 0$.

By C1 and the continuity of F, $\overline{\Psi}$ is continuous. By C3,

$$\overline{\Psi}(0, c^*) = -\int_0^{c^*} q(0, c_i)\, dF(c_i) - \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q_i(0, \widehat{c})\, dF(\widehat{c}), c_i\right) dF(c_i) < 0$$

By C2,

$$\begin{aligned}
\overline{\Psi}(q(0,0), c^*) &= q(0,0) - \int_0^{c^*} q(q(0,0), c_i)\, dF(c_i) \\
&- \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)}\int_0^{c^*} q(q(0,0), \widehat{c})\, dF(\widehat{c}), c_i\right) dF(c_i) > 0
\end{aligned}$$

If $Q'_{-i} > Q_{-i}$, then

$$\overline{\Psi}(Q'_{-i}, c^*) - \overline{\Psi}(Q_{-i}, c^*) = Q'_{-i} - Q_{-i} - \int_0^{c^*} \left(q\left(Q'_{-i}, c_i\right) - q\left(Q_{-i}, c_i\right)\right) dF\left(c_i\right)$$
$$- \int_{c^*}^{\infty} \left(q\left(\frac{1}{F\left(c^*\right)} \int_0^{c^*} q\left(Q'_{-i}, \widehat{c}\right) dF\left(\widehat{c}\right), c_i\right) - q\left(\frac{1}{F\left(c^*\right)} \int_0^{c^*} q\left(Q_{-i}, \widehat{c}\right) dF\left(\widehat{c}\right), c_i\right)\right) dF\left(c_i\right)$$
$$\geq Q'_{-i} - Q_{-i} - (1-\delta) \int_{c^*}^{\infty} \left(\frac{1}{F(c^*)} \int_0^{c^*} \left(q\left(Q_{-i}, \widehat{c}\right) - q\left(Q'_{-i}, \widehat{c}\right)\right) dF(\widehat{c})\right) dF(c_i)$$
$$= Q'_{-i} - Q_{-i} - (1-\delta) \frac{1 - F(c^*)}{F(c^*)} \int_0^{c^*} \left(q\left(Q_{-i}, \widehat{c}\right) - q\left(Q'_{-i}, \widehat{c}\right)\right) dF(\widehat{c})$$
$$\geq Q'_{-i} - Q_{-i} - (1-\delta)^2 (1 - F(c^*))(Q'_{-i} - Q_{-i})$$
$$= (Q'_{-i} - Q_{-i})(1 - (1-\delta)^2(1 - F(c^*))) > 0$$

where the inequalities follow from C2. Therefore for every c^* there exists a unique $Q^L\left(c^*\right) \in (0, q(0,0))$ such that $\overline{\Psi}\left(Q^L\left(c^*\right), c^*\right) = 0$, and a unique $Q^{H1}\left(c^*\right)$ defined by $Q^{H1}\left(c^*\right) = \frac{1}{F(c^*)} \int_0^{c^*} q\left(Q^L\left(c^*\right), c_i\right) dF(c_i)$. The functions $Q^L\left(c^*\right)$ and $Q^{H1}\left(c^*\right)$ are continuous by Theorem 2.1 in Jittorntrum (1978).

Next we show that $Q^L\left(c^*\right) \leq Q^{H1}\left(c^*\right)$. If $Q^L\left(c^*\right) > Q^{H1}\left(c^*\right)$, then

$$Q^L\left(c^*\right) - Q^{H1}\left(c^*\right) = \int_{c^*}^{\infty} q\left(Q^{H1}\left(c^*\right), c_i\right) dF\left(c_i\right) - \frac{1 - F\left(c^*\right)}{F\left(c^*\right)} \int_0^{c^*} q\left(Q^L\left(c^*\right), c_i\right) dF\left(c_i\right)$$
$$\leq \left(1 - F\left(c^*\right)\right) \left(q\left(Q^{H1}\left(c^*\right), c^*\right) - q\left(Q^L\left(c^*\right), c^*\right)\right) < \left(1 - F\left(c^*\right)\right) \left(Q^L\left(c^*\right) - Q^{H1}\left(c^*\right)\right)$$

which is a contradiction (the inequalities follow from C2).

Next, note that the function $\frac{1}{F(c)} \int_0^c q\left(Q^L, c_i\right) dF\left(c_i\right)$ decreases in c for every Q^L. Indeed, if $\tilde{c}^* < c^*$, then

$$\frac{1}{F\left(c^*\right)} \int_0^{c^*} q\left(Q^L, c_i\right) dF\left(c_i\right) - \frac{1}{F\left(\tilde{c}^*\right)} \int_0^{\tilde{c}^*} q\left(Q^L, c_i\right) dF\left(c_i\right) \tag{8}$$
$$= \frac{1}{F\left(c^*\right)} \int_{\tilde{c}^*}^{c^*} q\left(Q^L, c_i\right) dF\left(c_i\right) - \frac{F(c^*) - F(\tilde{c}^*)}{F\left(c^*\right) F\left(\tilde{c}^*\right)} \int_0^{\tilde{c}^*} q\left(Q^L, c_i\right) dF\left(c_i\right)$$
$$\leq \frac{F(c^*) - F(\tilde{c}^*)}{F\left(c^*\right)} q\left(Q^L, \tilde{c}^*\right) - \frac{F(c^*) - F(\tilde{c}^*)}{F\left(c^*\right)} q\left(Q^L, \tilde{c}^*\right) = 0$$

where the inequality follows from C2.

Let us now show that $Q^L(c^*)$ is increasing in c^*. Suppose that $\tilde{c}^* < c^*$ and $Q^L(\tilde{c}^*) > Q^L(c^*)$. Then $\overline{\Psi}(Q^L(\tilde{c}^*), c^*) > \overline{\Psi}(Q^L(c^*), c^*)$, because $\overline{\Psi}$ is strictly increasing in Q^L. Since $\overline{\Psi}(Q^L(c^*), c^*) = 0$ and $\overline{\Psi}(Q^L(\tilde{c}^*), \tilde{c}^*) = 0$, we get

$$\begin{aligned} & 0 < \overline{\Psi}(Q^L(\tilde{c}^*), c^*) - \overline{\Psi}(Q^L(\tilde{c}^*), \tilde{c}^*) \qquad (9) \\ = & \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF(c_i) + \int_{\tilde{c}^*}^{\infty} q\left(Q^{H1}(\tilde{c}^*), c_i\right) dF(c_i) \\ & - \int_0^{c^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF(c_i) - \int_{c^*}^{\infty} q\left(\frac{1}{F(c^*)} \int_0^{c^*} q\left(Q^L(\tilde{c}^*), \widehat{c}\right) dF(\widehat{c}), c_i\right) dF(c_i) \\ \leq & -\int_{\tilde{c}^*}^{c^*} \left(q\left(Q^L(\tilde{c}^*), c_i\right) - q\left(Q^{H1}(\tilde{c}^*), c_i\right)\right) dF(c_i) \leq 0 \end{aligned}$$

where the second inequality follows from C2, (8), and definition of Q^{H1}; the third inequality follows from $\tilde{c}^* < c^*$, $Q^L(\tilde{c}^*) \leq Q^{H1}(\tilde{c}^*)$ and C2. Hence we get a contradiction. Therefore, $Q^L(\tilde{c}^*) \leq Q^L(c^*)$, and

$$\begin{aligned} Q^{H1}(c^*) - Q^{H1}(\tilde{c}^*) &= \frac{1}{F(c^*)} \int_0^{c^*} q\left(Q^L(c^*), c_i\right) dF(c_i) - \frac{1}{F(\tilde{c}^*)} \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF(c_i) \\ &\leq \frac{1}{F(\tilde{c}^*)} \int_0^{\tilde{c}^*} q\left(Q^L(c^*), c_i\right) dF(c_i) - \frac{1}{F(\tilde{c}^*)} \int_0^{\tilde{c}^*} q\left(Q^L(\tilde{c}^*), c_i\right) dF(c_i) \leq 0 \end{aligned}$$

where the first inequality follows from (8), and the second from $Q^L(\tilde{c}^*) \leq Q^L(c^*)$ and C2. This proves that $Q^{H1}(c^*)$ is decreasing in c^*.

Next,

$$Q^{H1}(0) = q\left(Q^L(0), 0\right) \leq q(0, 0)$$

by C2, and therefore

$$q\left(Q^{H1}(0), 0\right) \geq q\left(q(0, 0), 0\right) > 0$$

where the first inequality is by C2 and the second by C3. Therefore, by C1 and the fact that $f > 0$,

$$Q^L(0) = \int_0^\infty q\left(Q^{H1}(0), c_i\right) dF(c_i) > 0$$

Finally, $\lim_{c^* \to \infty} Q^L(c^*) = \lim_{c^* \to \infty} Q^{H1}(c^*) = Q^{NC}$ by the definitions of $Q^L(c^*)$, $Q^{H1}(c^*)$ and Q^{NC}. ■

Proof of Lemma 8.

By the Envelope Theorem,

$$\Delta\Pi(c_i; c^*) = \beta\left(F(c^*)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c_i)\, dq_{-i} - (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i}, c_i)\, dq_{-i}\right)$$
$$= \beta\left(\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c_i)\, dq_{-i} - (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c_i)\, dq_{-i}\right)$$

Suppose first that

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} = (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} = 0$$

Then either $Q^L(c^*) = Q^{H1}(c^*) = Q^{H2}(c^*)$ or $\forall c' \geq c$, $\forall q_{-i} > \min\left\{Q^L(c^*), Q^{H2}(c^*)\right\}$, $q(q_{-i}, c') = 0$. In either case, $\Delta\Pi(c'; c^*) = 0$, $\forall c' \geq c$.

Suppose next that

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} = (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} \neq 0$$

Since $Q^{H1}(c^*) \geq Q^L(c^*)$ (Lemma 7), we have

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} = (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)\, dq_{-i} > 0$$

This in turn implies $Q^L(c^*) < Q^{H1}(c^*)$, $q\left(Q^L(c^*), c\right) > 0$ and (since $q(q_{-i}, c) \geq 0$) $Q^L(c^*) > Q^{H2}(c^*)$.

Let $Q(c) = \min\{q_{-i} \geq 0 : q(q_{-i}, c) = 0\}$; $Q(c) > 0$, because $q\left(Q^L(c^*), c\right) > 0$. The value of $Q(c)$ is determined by the first-order condition: $Q(c) = \frac{1}{\beta}\left(\rho(0) - \frac{\partial C(0,c)}{\partial q_i}\right)$. The function $Q(c)$ is differentiable and decreasing in c. The fact that $q\left(Q^L\left(c^*\right), c\right) > 0$ implies that $Q^L\left(c^*\right) < Q(c)$. Finally, by the definition of $Q(c)$, $\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c)dq_{-i} = \int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} q(q_{-i}, c)dq_{-i}$.

Condition C5 implies that for $q_{-i} \in \left(Q^L\left(c^*\right), Q(c)\right)$,

$$\frac{\partial q\left(q_{-i}, c\right)}{\partial c} < \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} q\left(q_{-i}, c\right) \tag{10}$$

Equation (10) implies

$$\int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} < \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \int_{Q^L(c^*)}^{\min\left\{Q(c), Q^{H1}(c^*)\right\}} q\left(q_{-i}, c\right) dq_{-i} \tag{11}$$

Since $q\left(Q^L\left(c^*\right), c\right) > 0$ and $q(q_{-i}, c)$ is decreasing in q_{-i}, we have $q\left(q_{-i}, c\right) > 0$, $\forall q_{-i} \in \left[Q^{H2}\left(c^*\right), Q^L\left(c^*\right)\right)$. Therefore, by C5, $\frac{\partial q(q_{-i}, c)}{\partial c} > \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q(Q^L(c^*), c)} q\left(q_{-i}, c\right)$ for every $q_{-i} \in \left[Q^{H2}\left(c^*\right), Q^L\left(c^*\right)\right)$, and thus

$$\int_{Q^{H2}(c^*)}^{Q^L(c^*)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} > \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \int_{Q^{H2}(c^*)}^{Q^L(c^*)} q\left(q_{-i}, c\right) dq_{-i} \tag{12}$$

Suppose first that $Q(c) < Q^{H1}(c^*)$. Then equations (11) and (12) and the fact that $q(Q(c), c) = 0$ imply

$$\begin{aligned} \frac{\partial \Delta\Pi\left(c; c^*\right)}{\partial c} &= \beta F\left(c^*\right) \int_{Q^L(c^*)}^{Q(c)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} + \beta F(c^*) \frac{dQ(c)}{dc} q(Q(c), c) \\ &- \beta\left(1 - F\left(c^*\right)\right) \int_{Q^{H2}(c^*)}^{Q^L(c^*)} \frac{\partial q\left(q_{-i}, c\right)}{\partial c} dq_{-i} \\ &< \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \beta\left(F\left(c^*\right) \int_{Q^L(c^*)}^{Q(c)} q\left(q_{-i}, c\right) dq_{-i} - \left(1 - F\left(c^*\right)\right) \int_{Q^{H2}(c^*)}^{Q^L(c^*)} q\left(q_{-i}, c\right) dq_{-i}\right) \\ &= \frac{\frac{\partial q\left(Q^L(c^*), c\right)}{\partial c}}{q\left(Q^L\left(c^*\right), c\right)} \Delta\Pi\left(c; c^*\right) = 0 \end{aligned} \tag{13}$$

Now suppose that $Q(c) > Q^{H1}(c^*)$. Then equations (11) and (12) imply

$$\frac{\partial \Delta\Pi(c;c^*)}{\partial c} = \beta\left(F(c^*)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} \frac{\partial q(q_{-i},c)}{\partial c} dq_{-i} - (1-F(c^*))\int_{Q^{H2}(c^*)}^{Q^L(c^*)} \frac{\partial q(q_{-i},c)}{\partial c} dq_{-i}\right) \tag{14}$$

$$< \frac{\frac{\partial q\left(Q^L(c^*),c\right)}{\partial c}}{q\left(Q^L(c^*),c\right)}\beta\left(F(c^*)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i},c)\,dq_{-i} - (1-F(c^*))\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i},c)\,dq_{-i}\right)$$

$$= \frac{\frac{\partial q\left(Q^L(c^*),c\right)}{\partial c}}{q\left(Q^L(c^*),c\right)}\Delta\Pi(c;c^*) = 0$$

Finally, suppose that $Q(c) = Q^{H1}(c^*)$. Then $\frac{\partial \Delta\Pi(c_+;c^*)}{\partial c}$ is given by the first line in (13), and $\frac{\partial \Delta\Pi(c_-;c^*)}{\partial c}$ is given by the first line in (14). Since $q(Q(c),c) = 0$ and $Q(c) = Q^{H1}(c^*)$, we have $\frac{\partial \Delta\Pi(c_+;c^*)}{\partial c} = \frac{\partial \Delta\Pi(c_-;c^*)}{\partial c} < 0$. ■

Proof of Lemma 9.

First, we will prove that there exists $\eta > 0$ such that for every $c_i \in [0,\overline{c}]$ and every $q_{-i} \leq q(0,0)$

$$q\left(q'_{-i},c_i\right) \geq q(q_{-i},c_i) + \eta q(q_{-i},c_i)\left(q_{-i} - q'_{-i}\right) \qquad \forall q'_{-i} \in (0,q_{-i}). \tag{15}$$

Let $\eta = \inf\left\{-\frac{\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}}{q(\widetilde{q}_{-i},0)} \mid \widetilde{q}_{-i} \in [0,q(0,0)]\right\}$. It is well defined since, by C3, $q(\widetilde{q}_{-i},0) > 0$ for every $\widetilde{q}_{-i} \in [0,q(0,0)]$, and $\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}$ is continuous by C1. By C2, $\eta > 0$.

If $q(q_{-i},c_i) = 0$, then (15) clearly holds. If $q(q_{-i},c_i) > 0$, then, by C2, $q(\widetilde{q}_{-i},c_i) > 0$ for every $\widetilde{q}_{-i} \in [0,q_{-i}]$. By C5,

$$\frac{\frac{\partial q(\widetilde{q}_{-i},c_i)}{\partial q_{-i}}}{q(\widetilde{q}_{-i},c_i)} < \frac{\frac{\partial q(\widetilde{q}_{-i},0)}{\partial q_{-i}}}{q(\widetilde{q}_{-i},0)} \leq -\eta$$

Thus for every $q'_{-i} \in (0,q_{-i})$,

$$q(q_{-i},c_i) - q\left(q'_{-i},c_i\right) = \int_{q'_{-i}}^{q_{-i}} \frac{\partial q(\widetilde{q}_{-i},c_i)}{\partial q_{-i}} d\widetilde{q}_{-i} \leq -\eta q(q_{-i},c_i)\left(q_{-i} - q'_{-i}\right)$$

and therefore (15) holds.

Next, we will prove that if $Q^L(c^*) \geq Q^{H2}(c^*)$, then

$$\Delta\Pi(c^*; c^*) \leq \beta q\left(Q^L(c^*), c^*\right)(1 - F(c^*))\left(Q^{H2}(c^*) - \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right) \quad (16)$$

where $\eta > 0$ satisfies (15).

Since $Q^L(c^*) \leq q(0,0)$, equation (15) implies that for every $q_{-i} \in \left[Q^{H2}(c^*), Q^L(c^*)\right]$,

$$q(q_{-i}, c^*) \geq q\left(Q^L(c^*), c^*\right) + \eta q\left(Q^L(c^*), c^*\right)\left(Q^L(c^*) - q_{-i}\right)$$

Therefore

$$\begin{aligned}\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i}, c^*)\, dq_{-i} &\geq q\left(Q^L(c^*), c^*\right)\int_{Q^{H2}(c^*)}^{Q^L(c^*)}\left(1 + \eta\left(Q^L(c^*) - q_{-i}\right)\right) dq_{-i} \\ &= q\left(Q^L(c^*), c^*\right)\left(\left(Q^L(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right)\end{aligned} \quad (17)$$

For every $q_{-i} \in \left[Q^L(c^*), Q^{H1}(c^*)\right]$, $q(q_{-i}, c^*) \leq q\left(Q^L(c^*), c^*\right)$, and thus

$$\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c^*)\, dq_{-i} \leq q\left(Q^L(c^*), c^*\right)\left(Q^{H1}(c^*) - Q^L(c^*)\right) \quad (18)$$

Equations (17) and (18) imply

$$\begin{aligned}&\Delta\Pi(c^*; c^*) = \beta\left(F(c^*)\int_{Q^L(c^*)}^{Q^{H1}(c^*)} q(q_{-i}, c^*)\, dq_{-i} - (1 - F(c^*))\int_{Q^{H2}(c^*)}^{Q^L(c^*)} q(q_{-i}, c^*)\, dq_{-i}\right) \\ &\leq \beta\left(\begin{array}{c} F(c^*)\, q\left(Q^L(c^*), c^*\right)\left(Q^{H1}(c^*) - Q^L(c^*)\right) \\ -(1 - F(c^*))q\left(Q^L(c^*), c^*\right)\left(\left(Q^L(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right)\end{array}\right) \\ &= \beta q\left(Q^L(c^*), c^*\right)\left(\begin{array}{c}\left(Q^{H1}(c^*) - Q^L(c^*)\right) \\ -(1 - F(c^*))\left(\left(Q^{H1}(c^*) - Q^{H2}(c^*)\right) + \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2\right)\end{array}\right)\end{aligned}$$

Note that by definition of $Q^{H1}(c^*)$ and $Q^L(c^*)$,

$$\begin{aligned} Q^{H1}(c^*) - Q^L(c^*) &= (1 - F(c^*)) \left(Q^{H1}(c^*) - \frac{1}{1 - F(c^*)} \int_{c^*}^{\infty} q\left(Q^{H1}(c^*), c_i\right) dF(c_i) \right) \\ &\leq (1 - F(c^*)) Q^{H1}(c^*) \end{aligned}$$

Thus

$$\Delta\Pi(c^*; c^*) \leq \beta q\left(Q^L(c^*), c^*\right)(1 - F(c^*)) \left(Q^{H2}(c^*) - \frac{\eta}{2}\left(Q^L(c^*) - Q^{H2}(c^*)\right)^2 \right)$$

Finally, let $\widehat{c} > 0$ be such that $q_i(0, \widehat{c}) \leq \left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}} \right)^2$, where $\eta > 0$ satisfies condition (15) (such $\widehat{c}$ exists by C6 and the fact that $Q^L(0) > 0$ by Lemma 7). We will prove that if $F(\widehat{c}) < 1$, then there exists $c^* \in (0, \overline{c})$ such that the "min" mechanism with threshold c^* is incentive compatible.

By Lemma 8, it is enough to show that there exists $c^* \in (0, \overline{c})$ such that $\Delta\Pi(c^*; c^*) = 0$.

Note that $\Delta\Pi(c_i; c^*)$ is continuous in c_i and c^* (since Π_i is continuous in (q_{-i}, c_i), c_i is continuously distributed, and $Q^L(c^*)$, $Q^{H1}(c^*)$, and $Q^{H2}(c^*)$ are continuous in c^* (Lemmas 6 and 7)). Thus it is enough to show that $\Delta\Pi(0; 0) > 0$, and $\Delta\Pi(c^*; c^*) \leq 0$ for some $c^* \in (0, \overline{c})$.

By Lemmas 6 and 7, $Q^{H2}(0) = Q^{NC} > Q^L(0)$. By C2 and C3, $q\left(Q^L(0), 0\right) \geq q(q(0, 0), 0) > 0$. Therefore

$$\Delta\Pi(0; 0) = \Pi_i\left(Q^L(0), 0\right) - \Pi_i\left(Q^{H2}(0), 0\right) = \beta \int_{Q^L(0)}^{Q^{H2}(0)} q(q_{-i}, 0)\, dq_{-i} > 0$$

If $q\left(Q^L(\widehat{c}), \widehat{c}\right) = 0$, then $\Pi_i\left(Q^L(\widehat{c}), \widehat{c}\right) = 0$, and thus $\Delta\Pi(\widehat{c}; \widehat{c}) \leq 0$.

Suppose that $q\left(Q^L(\widehat{c}), \widehat{c}\right) > 0$. Note that $Q^L(\widehat{c}) \geq Q^L(0)$ (Lemma 7), and $Q^{H2}(\widehat{c}) \leq q(0, \widehat{c}) \leq \left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}} \right)^2$ by C2.

Thus

$$Q^L(\widehat{c}) - Q^{H2}(\widehat{c}) \geq Q^L(0) - \left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right)^2 = \sqrt{\frac{2}{\eta}}\left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right) > 0$$

Therefore, by inequality (16) we get

$$\begin{aligned}
&\Delta\Pi(\widehat{c};\widehat{c}) \leq \beta q\left(Q^L(\widehat{c}), \widehat{c}\right)(1 - F(\widehat{c}))\left(Q^{H2}(\widehat{c}) - \frac{\eta}{2}\left(Q^L(\widehat{c}) - Q^{H2}(\widehat{c})\right)^2\right) \\
&\leq \beta q\left(Q^L(\widehat{c}), \widehat{c}\right)(1 - F(\widehat{c}))\left(\left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right)^2 - \frac{\eta}{2}\left(\sqrt{\frac{2}{\eta}}\left(\sqrt{\frac{1}{2\eta} + Q^L(0)} - \sqrt{\frac{1}{2\eta}}\right)\right)^2\right) \\
&= 0
\end{aligned}$$

■

Calculations for Example 1. Suppose that $\beta = \gamma = 1$ and $c_i \sim U[0, \overline{c}]$. Then the values of $Q^L(c^*)$, $Q^{H1}(c^*)$ and $Q^{H2}(c^*)$, as defined by Lemmas 6 and 7, are

$$\begin{aligned}
Q^L(c^*) &= \frac{1}{3}\left(K - \frac{\overline{c}}{2}\right) - \frac{(\overline{c} - c^*)^2}{6(\overline{c} + c^*)}; \\
Q^{H1}(c^*) &= \frac{1}{3}\left(K - \frac{\overline{c}}{2}\right) + \frac{(2\overline{c} + c^*)(\overline{c} - c^*)}{6(\overline{c} + c^*)}; \\
Q^{H2}(c^*) &= \frac{1}{3}\left(K - \frac{\overline{c}}{2}\right) - \frac{c^*}{6}
\end{aligned}$$

Lemma 8 implies that the "min" mechanism with threshold c^* is incentive compatible if and only if $\Delta\Pi(c^*; c^*) = 0$. In this case, substituting the above expressions into the definition of $\Delta\Pi(c^*; c^*)$ and equating to zero results in

$$K = \frac{3c^*}{2} - \frac{\overline{c}}{4} - \frac{2(c^*)^2 - 7c^*\overline{c} + \overline{c}^2}{8(c^* + \overline{c})}$$

Let $c^*(K)$ be the value of c^* that solves this equation; then $c^*(K)$ increases in K (because the right-hand side is strictly increasing in c^*) and reaches $\overline{c}$ when $K = \frac{3}{2}\overline{c}$. Therefore an incentive compatible "min" mechanism exists whenever $K < \frac{3}{2}\overline{c}$.

Lemmas 6 and 7 imply that every type's output is strictly positive under the "min" mechanism with threshold c^* if and only if $q(Q^{H1}(c^*), \overline{c}) > 0$. If $K = \frac{3}{2}\overline{c}$, then $c^*(K) = \overline{c}$ and $Q^{H1}(c^*) = \frac{\overline{c}}{3} = Q^{NC}$, so $q(Q^{H1}(c^*), \overline{c}) = q(Q^{NC}, \overline{c}) = \frac{1}{2}\left(K - \frac{\overline{c}}{3} - \overline{c}\right) = \frac{\overline{c}}{12} > 0$. By continuity of $c^*(K)$, $Q^{H1}(c^*)$ and $q(q_{-i}, c_i)$, this implies that $q(Q^{H1}(c^*), \overline{c}) > 0$ if K is close enough to $\frac{3}{2}\overline{c}$. ■

References

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