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Maximal loss from collusion in IPV symmetric auctions

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Abstract

We derive a bound on the seller's revenue loss in optimal auctions from unanticipated bidder collusion. The relative loss is rather small when there are few bidders. It is increasing with the number of bidders but at a slow rate.

Keywords: Collusion; Auctions

JEL classification: D44; L41

1. Introduction

The goal of this paper is to quantify the impact of bidder collusion on seller's revenue in auctions. There exists some empirical research estimating the revenue losses associated with collusion.[1] However, we are not aware of any theoretical treatment of this question except for McMillan (1991), whose contribution we discuss below. In this paper we perform a worst case scenario analysis of the effect of bidder collusion on the expected revenue at a single unit optimal auction in the symmetric private values environment.

Our analysis is the worst case approach in the following sense. First, the seller does not anticipate collusion and just offers some implementation of the optimal auction from a class of mechanisms containing most of the standard auction formats. Second, collusion between the bidders is assumed to be completely frictionless, and hence most detrimental to seller. Finally, we focus on the computation of the highest possible relative revenue loss within a class of distributions of the valuations which satisfy the standard nondecreasing hazard rate property.

The upper bound on the relative revenue loss from collusion is rather modest for a small number of bidders. This bound increases in the number of bidders and asymptotically approaches 100%, but the convergence is slow. We also show that the revenue loss may be higher when the hazard rate property is not satisfied.

This paper belongs to a small but growing literature on the assessment of the robustness of certain simple mechanisms: Neeman (2003) studies the effectiveness of English auctions, Rogerson (2003) studies the performance of simple procurement mechanisms, McAfee (2002) evaluates the effectiveness of simple matching schemes. The analysis in McMillan (1991) is most closely related to our paper. McMillan (1991) has a simple model for evaluating the government's relative loss from collusion in procurement auctions. He assumes that the bidders' costs are independently uniformly distributed on an arbitrary nonnegative support, while our results are for a class of independent distributions which satisfy the nondecreasing hazard rate property. Also he assumes that the government does not optimize with respect to a ceiling price, while we assume that the seller optimally chooses the reservation price. McMillan (1991) derives a formula for the expected difference between collusive and competitive prices as a percentage of collusive price. This measure of the loss from collusion is increasing in the spread of the possible costs and in the number of bidders. When the number of bidders is n, the relative loss is at most $\frac{n-1}{n+1}$.

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[1] For example, Baldwin et al. (1997) study the bidding behavior in U.S. Forest Timber Sales and find the revenue loss from collusion to be 7.9%. Froeb et al. (1993) study the effect of the bidders' collusion in U.S. Department of Defense procurement auctions for frozen fish and find the average price increase due to collusion to be 27.3%.

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2. The model and the main result

There is one seller who owns a single indivisible good and $n \geq 2$ bidders.[2] Each bidder i has a valuation θ_i for the good which is known only to him. Valuations are identically and independently distributed according to a continuous cumulative distribution function F and everywhere positive density f with support $[\underline{\theta},\overline{\theta}]$, where $0 \leq \underline{\theta} < \overline{\theta} \leq +\infty$. This distribution is common knowledge. Define the hazard rate $h(\theta) = \frac{f(\theta)}{1-F(\theta)}$. For the main result we require the distribution to satisfy a standard hazard rate condition.

Condition 1. (Nondecreasing hazard rate) $h(\theta') \geq h(\theta)$ for every $\theta' > \theta$.

All players have quasi-linear utilities. Bidder i's utility is $\theta_i p_i - t_i$, when p_i is his probability of getting the good and t_i is his payment. The seller's utility is $\sum_i t_i$.

Next we find formulas for the seller's expected revenue without collusion and with collusion for a given distribution of valuations and number of bidders. After that we establish an upper bound on the maximal relative revenue loss from unanticipated collusion in the class of distributions with nondecreasing hazard rate (Theorem 1). Finally, we show that this bound is tight (Example 1), and also provide an example of a distribution which violates Condition 1 and where the revenue loss is higher than the upper bound in the main result (Example 2).

We begin with restating the standard results on the optimal auction without collusion.

Proposition 1. *Assume $\theta - \frac{1}{h(\theta)}$ is strictly increasing. Then*

(i) Every mechanism which implements the optimal auction must have the following allocation conditional on the bidders' types for almost every $(\theta_1 \ldots, \theta_n)$:

$$p_i(\theta_1, \ldots, \theta_n) = \begin{cases} 1 & \text{if } \theta_i > \max\{\max_{j \neq i} \theta_j, R\} \\ 0 & \text{otherwise} \end{cases} \text{ for every } i,$$

where $R = \inf\left\{\theta \in [\underline{\theta},\overline{\theta}] : \theta - \frac{1}{h(\theta)} \geq 0\right\}$

(ii) The expected revenue from the optimal mechanism is

$$R(1 - F^n(R)) + \int_R^{\overline{\theta}} \left(1 - F_{(2,n)}\left(\tilde{\theta}\right)\right) d\tilde{\theta},$$

where $F_{(2,n)}(\theta)$ is the distribution of the second order statistic.[3]

Proof. See for example Chapters 2 and 5 in Krishna (2002). □

Notice that the assumption of nondecreasing hazard rate implies that $\theta - \frac{1}{h(\theta)}$ is strictly increasing, and thus the above characterization of the optimal auction applies to our setup. In what follows we denote by $\Pi_{n,F}^*$ the expected revenue from the optimal mechanism in the absence of collusion when there are n bidders and the distribution of the valuations is F.

Let us call an auction which implements the optimal mechanism as a noncooperative equilibrium to be a *standard auction* if it satisfies the following properties:

(i) the bidders can avoid any payments if they do not get the good;
(ii) the cheapest way for any bidder to obtain the good is to pay the reservation price R.

Notice that the first- or second-price auction implementations of the optimal mechanism (or their dynamic counterparts, Dutch and English auctions) are standard auctions.[4] The most efficient collusive scheme between the bidders is to buy the good from the seller at the price R when at least one of the bidders' valuations exceeds R, and not to buy otherwise. The expected revenue of the seller in this case is just $\Pi_{n,F} = R(1 - F^n(R))$.

The main result presents a bound on the relative revenue loss in standard auctions when the distribution satisfies Condition 1.

Theorem 1.

$$\frac{\Pi_{n,F}^* - \Pi_{n,F}}{\Pi_{n,F}^*} \leq \frac{\sum_{j=2}^{n} \frac{1}{j}}{\sum_{j=1}^{n} \frac{1}{j}}.$$

Proof. The idea of the proof is quite simple. We consider an arbitrary distribution F and replace the part of it starting from R with the exponential distribution with the parameter $h(R)$. This provides an upper bound on the relative loss from the presence of collusion when the reservation price is R and $\Pr\{\theta_i \leq R\} = F(R)$. Finally, we maximize the measure of the relative loss over R and $F(R)$.

Notice that for $\theta \geq R$ by Condition 1 we have $\int_{\underline{\theta}}^{\theta} h\left(\tilde{\theta}\right) d\tilde{\theta} \geq \int_{\underline{\theta}}^{R} h\left(\tilde{\theta}\right) d\tilde{\theta} + (\theta - R)h(R)$.

Denote $c = e^{-\int_{\underline{\theta}}^{R} h(\tilde{\theta}) d\tilde{\theta}}$ and $G(\theta) = 1 - ce^{-(\theta - R)h(R)}$ for $\theta \geq R$.

Notice that $F(\theta) = 1 - e^{-\int_{\underline{\theta}}^{\theta} h(\tilde{\theta}) d\tilde{\theta}} \geq 1 - ce^{-(\theta - R)h(R)} = G(\theta)$.

Observe that $1 - F_{(2,n)}(\theta) = 1 - nF^{n-1}(\theta) + (n-1)F^n(\theta) \leq 1 - nG^{n-1}(\theta) + (n-1)G^n(\theta)$, since a function $1 - nz^{n-1} + (n-1)z^n$ is decreasing in z for $z \in [0,1]$.

Also notice that $1 - nG^{n-1}(\theta) + (n-1)G^n(\theta) = (1 - G(\theta))\left(\sum_{i=0}^{n-1} G^i(\theta) - nG^{n-1}(\theta)\right)$.

[2] Throughout the paper we use feminine pronouns for the seller, and masculine pronouns for the bidders.

[3] $F_{(2,n)}(\theta) = F^n(\theta) + nF^{n-1}(\theta)(1 - F(\theta))$.

[4] There also exist other mechanisms which implement the optimal auction that do not satisfy these properties: for example, the all-pay auction with the reservation price $RF^{n-1}(R)$. We do not consider such mechanisms.

For every $i=0,\ldots,n-1$ we have

$$\int_R^{\overline{\theta}}\left(1-G\left(\tilde{\theta}\right)\right)G^i\left(\tilde{\theta}\right)d\tilde{\theta}$$
$$=\int_R^{\overline{\theta}}ce^{-\left(\tilde{\theta}-R\right)h(R)}\left(1-ce^{-\left(\tilde{\theta}-R\right)h(R)}\right)^i d\tilde{\theta}=$$
$$=\frac{1}{h(R)}\frac{1}{i+1}\left[\left(1-ce^{-\left(\tilde{\theta}-R\right)h(R)}\right)^{i+1}\right]_R^{\overline{\theta}}$$
$$=\frac{1}{h(R)}\frac{1}{i+1}\left[\left(1-ce^{-\left(\overline{\theta}-R\right)h(R)}\right)^{i+1}-(1-c)^{i+1}\right]=$$
$$=\frac{1}{h(R)}\frac{1}{i+1}\left(G^{i+1}\left(\overline{\theta}\right)-G^{i+1}(R)\right).$$

Summing up and changing $j=i+1$ we have

$$\int_R^{\overline{\theta}}\left(1-F_{(2,n)}(\theta)\right)d\tilde{\theta}\le\int_R^{\overline{\theta}}\left(1-G\left(\tilde{\theta}\right)\right)$$
$$\times\left(\sum_{j=1}^{n}G^{j-1}\left(\tilde{\theta}\right)-nG^{n-1}\left(\tilde{\theta}\right)\right)d\tilde{\theta}=$$
$$=\frac{1}{h(R)}\left[\sum_{j=1}^{n}\tfrac{1}{j}\left(G^j\left(\overline{\theta}\right)-G^j(R)\right)-\left(G^n\left(\overline{\theta}\right)-G^n(R)\right)\right]$$
$$\le R\left[\sum_{j=1}^{n}\tfrac{1}{j}\left(1-(1-c)^j\right)-\left(1-(1-c)^n\right)\right].$$

The last inequality comes from the definition for R the fact that $G(R)=F(R)$, and since $\sum_{j=1}^{n}\frac{1}{j}y^j-y^n$ is increasing in y for $y\in[0,1]$.

Notice that $R(1-F^n(R))=R(1-(1-c^n))$, and thus $\Pi^*_{n,F}\le R\left[\sum_{j=1}^{n}\frac{1}{j}\left(1-(1-c)^j\right)\right]$.

Hence $\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}\le 1-\frac{(1-(1-c)^n)}{\sum_{j=1}^{n}\frac{1}{j}\left(1-(1-c)^j\right)}$.

Let us show that $\frac{\sum_{j=1}^{n}\frac{1}{j}\left(1-(1-c)^j\right)}{(1-(1-c)^n)}$ is decreasing in c. Notice that it suffices to show that $\frac{1-(1-c)^j}{1(1-c)^n}$ is decreasing in c for any j = 1,…,n. This is confirmed since $\frac{1-x^j}{1-x^n}$ is increasing in x for $x\in(0,1)$ and any $j=1,\ldots,n$.

Hence $\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}\le 1-\frac{1}{\sum_{j=1}^{n}\frac{1}{j}}=\frac{\sum_{j=2}^{n}\frac{1}{j}}{\sum_{j=1}^{n}\frac{1}{j}}$. □

Next we demonstrate that the obtained bound is tight.

Example 1. Consider $F(\theta)=\begin{cases}0 & \text{if } \theta\in\left(-\infty,\frac{1}{\lambda}\right)\\ 1-e^{-\lambda\theta+1} & \text{if } \theta\in\left[\frac{1}{\lambda},+\infty\right)\end{cases}$ where $\lambda>0$. Then $R=\frac{1}{\lambda}$ and F(R)=0. Following the steps of the above derivation weak inequalities hold as equalities throughout the argument. Hence $\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}=\frac{\sum_{j=2}^{n}\frac{1}{j}}{\sum_{j=1}^{n}\frac{1}{j}}$.

When Condition 1 is not satisfied the maximal loss can be higher. This is demonstrated in the next example.

Example 2. Let $n=2$ and $F(\theta)=\begin{cases}0 & \text{if } \theta\in(-\infty,\underline{\theta})\\ 1-4\left(1+\frac{\theta}{\underline{\theta}}\right)^{-2} & \text{if } \theta\in[\underline{\theta},+\infty),\end{cases}$ where $\underline{\theta}>0$.

Notice that $h(\theta)=\frac{2}{\theta+\underline{\theta}}$ is decreasing, while $\theta-\frac{1}{h(\theta)}=\frac{1}{2}(\theta-\underline{\theta})$ is increasing, and thus Proposition 1 applies. Also observe that $R=\underline{\theta}$.

Hence $\Pi_{n,F}=R(1-F^2(R))=\underline{\theta}$, and $\Pi^*_{n,F}-\Pi_{n,F}=\int_{\underline{\theta}}^{+\infty}\left(1-F\left(\tilde{\theta}\right)\right)^2d\tilde{\theta}=16\int_{\underline{\theta}}^{+\infty}\left(1+\frac{\tilde{\theta}}{\underline{\theta}}\right)^{-4}d\tilde{\theta}=\frac{2}{3}\underline{\theta}$.

Thus $\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}=\frac{2}{5}>\frac{1}{3}$.

3. Discussion

Let us consider how the maximal relative loss from the presence of collusion depends on the number of bidders.

n	2	3	4	5	10	100	∞
$\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}\lesssim$	0.33	0.45	0.52	0.56	0.66	0.81	1.00

Quite remarkably for a small number of bidders the maximal relative loss is not as high as one could expect. The intuition for this is as follows. The revenue from the optimal auction can be decomposed into two parts: a "monopoly component" $R(1-F^n(R))$ which collects a reservation price R from the winning bidder, and an "auction component" $\int_R^{\overline{\theta}}\left(1-F_{(2,n)}\left(\tilde{\theta}\right)\right)d\tilde{\theta}$ which exploits the competition among the bidders. It turns out that the "monopoly component" accounts for a large share of the revenue when the number of bidders is small. As the number of the bidders grows, the "auction component" becomes relatively more important. However, it can be noticed that the rate is quite slow.

By the divergence of the harmonic series the maximal loss reaches 100% as the number of bidders goes to infinity: $\lim_{n\to\infty}\frac{\sum_{j=2}^{n}\frac{1}{j}}{\sum_{j=1}^{n}\frac{1}{j}}=1$. Notice though that an unbounded support is required for this to happen. In case the upper bound of the support $\underline{\theta}$ is finite, the maximal loss is $\frac{\overline{\theta}-R}{\overline{\theta}}<1$.

For many standard distributions the loss is much smaller than the upper bound.

Example 3. (i) Let θ_i be uniformly distributed on [0,1]. Then

n	2	3	4	5	10	100	∞
$\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}\lesssim$	0.10	0.18	0.23	0.28	0.39	0.49	0.50

(ii) Let θ_i be uniformly distributed on [5,6]. Then

n	2	3	4	5	10	100	∞
$\frac{\Pi^*_{n,F}-\Pi_{n,F}}{\Pi^*_{n,F}}\lesssim$	0.06	0.09	0.11	0.12	0.14	0.16	0.17

There seems to be a common feeling among the economists that collusion may have devastating results on revenue. The results of this paper suggest that in many cases the loss from collusion in optimal auctions just cannot be large.[5]

[5] The revenue loss from collusion can be much larger in the efficient auction, but in that case it can be argued that the seller should not be concerned about the revenue.

Of course it remains to evaluate to what extent the nondecreasing hazard rate condition is a property of economically relevant distributions, rather than just a convenient modeling assumption. Also we need to obtain the estimates of the loss from collusion when the bidders are asymmetric, when the valuations are interdependent, and when there are multiple units for sale.

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