Selling two units of a customizable good*

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Abstract

A seller can produce two units of a good. Each unit can be customized into either product a or product b, and buyers privately learn their valuations for each product. First I consider the case when the second unit is more costly to produce than the first. Under certain distributional conditions the search for the optimal mechanism can be restricted to a class where there is no uncertainty about the number of units the buyer will receive, i.e. the buyer chooses whether to get 0, 1, or 2 units. However, there still may be uncertainty over which product the buyer will get. Buyers whose valuation for their favorite product is high, both in absolute terms and relative to the other product, purchase two units of their favorite product with certainty. Buyers with low values are excluded from purchasing. Buyers with values in the intermediate range typically get a lottery over different products. I compare the fully optimal mechanism with the mechanism that optimally sells each unit separately and show that the solutions coincide when the fully optimal mechanism is deterministic. Another case I consider is when the cost of the second unit is not higher than the cost of the first. Many qualitative properties of the solution are similar to the previous case, but the key difference is that

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the optimal mechanisms only contain contracts such that the buyer chooses whether to get 0 or 2 units.

1 Introduction

A seller can produce two units of a good. Each unit can be customized into either product a or product b. Buyers privately learn their valuations for each product, and the valuations are continuously distributed. The objective of the paper is to find the optimal stochastic selling mechanisms for different cost structures of the seller.

There has been a significant recent progress in understanding the properties of the solution to the problem of selling multiple products when there is a single unit of each product available. See, for example, Daskalakis, Deckelbaum and Tzamos (2017) and references therein. These results can be easily extended to the case of multiple units of multiple products as long as the unit costs of each product remain constant. However the case of multiple units and multiple products with varying unit costs remains not well understood. Wilson (1993) and Armstrong (1996) studied a continuous quantities model with increasing marginal costs and explicitly solved some examples using the "integration along rays" technique. Yet Rochet and Chone (1998) argued that those examples are special and provided a general approach for numerically solving such problems. There had been limited progress since then, and the economics of such problems remains not very clear. In this paper I adapt existing techniques for the case of single units of multiple products (Pavlov 2011a, 2011b) to study a relatively simple model with two units of customizable good. The goal is to obtain explicit solutions and understand the involved tradeoffs. Hopefully the methods and insights of this paper can be used to study other multidimensional mechanism design problems as well.

In Section 3 I consider the case when the second unit is more costly to produce than the first. I show that under certain distributional conditions the search for the optimal mechanism can be restricted to a class where there is no uncertainty about the number of units the buyer will receive, i.e. the buyer chooses whether to get 0, 1, or 2 units. However, there still may be uncertainty over which product the buyer will get. Using this result, I simplify the two-dimensional mechanism problem design problem so that it can be solved by standard optimal control methods. There is no distortion for buyers whose valuation for their favorite product is high, both in absolute terms and relative to the other product. Such buyers purchase two units of their favorite product with certainty. Buyers with low values are excluded from purchasing. Solved examples suggest that the optimal mechanism typically contains only a few point contracts. Buyers with values in the intermediate range usually get a lottery over different products. I compare the fully optimal mechanism with the mechanism that optimally sells each unit separately and show that the solutions coincide when the fully optimal mechanism is deterministic.

In Section 4 I consider the case when the cost of the second unit is not higher than the cost of the first. In this case it is without loss of generality to consider mechanisms where the buyer chooses whether to get 0 or 2 units. Similarly to the previous case, there is no distortion at the top in a particular sense, buyers with low values are excluded, and in the solved examples the optimal mechanisms are simple. Buyers whose values are not too low and whose difference in the valuations is not too high usually get a lottery over different products.

2 Model

A seller can produce two units of a good. Production costs of the first and the second units are c_1 and c_2. Each unit can be customized at no cost into one of two possible versions which will be called *product* a and *product* b. A buyer values each unit of product a and b at v_a and v_b, respectively. The valuations are known only to the buyer, and they are continuously distributed according to a symmetric strictly positive continuously differentiable density f on $\Theta = [\underline{v}, \overline{v}]^2$, where $0 \leq \underline{v} < \overline{v}$.

The buyer's utility is $\sum_{i=a,b} \sum_{j=1,2} v_i p_{ij} - T$, where p_{ij} is the probability that the buyer receives jth unit that was customized into product i, and T is his payment to the seller. The seller's utility is $T - \sum_{i=a,b} \sum_{j=1,2} c_j p_{ij}$.

By the revelation principle we can without loss of generality assume that the seller offers a direct mechanism. It consists of a set Θ of type reports, an allocation rule $p_{ij} : \Theta \to [0,1]$ for $i = a, b,\ j = 1, 2$, and a payment rule $T : \Theta \to \mathbb{R}$.[1] The seller's problem is stated below. There are three kinds of constraints: feasibility (F), incentive compatibility (IC), and individual rationality (IR). A mechanism that satisfies all constraints is called *admissible.* Feasibility constraints consist of nonnegativity constraints on probabilities, requirements that the probability that each unit is produced is at most one, and requirements that ensure that it is never the case that unit 2 is produced but unit 1 is not.

Program I : $$\max_{p_{ij},T} E\left[T(v) - \sum_{i=a,b}\sum_{j=1,2} c_j p_{ij}(v)\right]$$

subject to

F: $p_{ij}(v) \geq 0$ for every $v \in \Theta$, $i = a, b$, $j = 1, 2$

$$\sum_{i=a,b} p_{ij}(v) \leq 1 \text{ for every } v \in \Theta,\ j = 1, 2$$

$$\sum_{i=a,b} p_{i1}(v) \geq \sum_{i=a,b} p_{i2}(v) \text{ for every } v \in \Theta$$

IC: $$\sum_{i=a,b}\sum_{j=1,2} v_i p_{ij}(v) - T(v) \geq \sum_{i=a,b}\sum_{j=1,2} v_i p_{ij}(v') - T(v') \text{ for every } v, v' \in \Theta$$

IR: $$\sum_{i=a,b}\sum_{j=1,2} v_i p_{ij}(v) - T(v) \geq 0 \text{ for every } v \in \Theta$$

Denote $U(v) := \sum_{i=a,b} \sum_{j=1,2} v_i p_{ij}(v) - T(v)$.

There are two qualitatively distinct cases depending on whether unit 1 or unit 2 is cheaper to produce. *Decreasing returns* case with $c_2 > c_1 \geq 0$ is considered in Section 3, *nondecreasing returns* case with $c_1 \geq c_2 \geq 0$ is in Section 4.

[1] The seller never benefits from randomized payments because the payoffs are linear in money. Thus there is no loss of generality in restricting attention to deterministic payment rules.

3 Decreasing returns

3.1 Simplifying the problem

In this case unit 2 is more expensive than unit 1: $c_2 > c_1 \geq 0$. Straightforward cost minimization logic implies that unit 2 will be produced with positive probability only if unit 1 is produced with probability 1.

Lemma 1 *Let $c_2 > c_1$. For every v, if $\sum_{i=a,b} p_{i2}(v) > 0$ then $\sum_{i=a,b} p_{i1}(v) = 1$.*[2]

Next we show that under a particular distributional assumption the search for optimal mechanisms can be restricted to the class where there is no uncertainty about the number of units the buyer will receive, i.e. depending on the report he knows whether he will get 0, 1, or 2 units. However, there still may be uncertainty over which product the buyer will get.

Proposition 1 *Let $c_2 > c_1$. Suppose*

$$3f(v) + \sum_{i=a,b} (v_i - c_j) \frac{\partial f(v)}{\partial v_i} \geq 0 \text{ and } c_j \leq \underline{v} \text{ for } j = 1, 2. \tag{1}$$

Then there is no loss for the seller in optimizing over mechanisms that satisfy

$$\sum_{i=a,b} \sum_{j=1,2} p_{ij}(v) \in \{0, 1, 2\} \text{ for every } v \in \Theta.$$

The second part of condition (1) requires the costs to be relatively low, and the first part demands that the density does not decrease too quickly. I conjecture that the result holds under more general conditions, but I did not investigate it here.

The idea of the proof is as follows. Each message in the mechanism leads to particular expected quantity of product a, $Q_a := \sum_{j=1,2} p_{aj}$, product b, $Q_b := \sum_{j=1,2} p_{bj}$, and payment, T. Consider all such point contracts achievable through the mechanism. For the sake of the

[2] The proofs are relegated to the Appendix unless stated otherwise. The proof of Lemma 1 is straightforward and omitted.

argument here assume that the resulting set contains all points such that $Q_a \geq 0$, $Q_b \geq 0$, $Q_a + Q_b \leq 2$, but the proof deals with a general case.

Next consider removing contracts such that $Q_a + Q_b \in (1, 2)$. High-value buyers that were previously choosing contracts that were removed will switch to contracts with $Q_a + Q_b = 2$. Such contracts are relatively more profitable because they are more expensive, and because the seller's cost is assumed to be sufficiently small. Low-value buyers will switch to the less profitable one-unit contracts with $Q_a + Q_b = 1$. The net effect on the profit depends on the ratio of high- and low-value buyers, and it is guaranteed to be positive by the distributional assumption that ensures that the density is not decreasing too quickly. Similarly it is profitable to remove contracts such that $Q_a + Q_b \in (0, 1)$, and let the buyers self-select into one-unit contracts with $Q_a + Q_b = 1$ and the null contract with $Q_a = Q_b = 0$.

Using Proposition 1 we can rewrite the buyer's payoff. Consider the payoff of the buyer of type $v = (v_a, v_b)$ such that $v_a \geq v_b$:

$$U(v) = \begin{cases} 0 & \text{if } \sum_{i=a,b} p_{ij}(v) = 0 \text{ for } j = 1, 2 \\ (v_a - v_b) p_{a1}(v) + v_b - T(v) & \text{if } \sum_{i=a,b} p_{i1}(v) = 1,\ \sum_{i=a,b} p_{i2}(v) = 0 \\ (v_a - v_b) \left(\sum_{j=1,2} p_{aj}(v) \right) + 2v_b - T(v) & \text{if } \sum_{i=a,b} p_{ij}(v) = 1 \text{ for } j = 1, 2 \end{cases}$$

We can think of the buyer making a choice in two steps. First he chooses whether to buy 0, 1, or 2 units of the good. Second he decides on the expected quantities of products a and b : p_{a1} and $1 - p_{a1}$ if he is buying one unit, $\sum_{j=1,2} p_{aj}$ and $2 - \sum_{j=1,2} p_{aj}$ if he is buying two units. The expression for the buyer's payoff given above suggests that the choice of the expected quantities of products a and b will be determined by how much more the buyer likes product a relative to product b, $v_a - v_b$. The initial choice of the number of units of the good to buy is determined by how high is the buyer's valuation for the less preferred good v_b, as well as the utility from optimally choosing the expected quantities of products a and b at the second step.

Next we reformulate the seller's problem based on this idea. In a symmetric environment there is no loss for the seller in restricting attention to symmetric mechanisms. Thus we can state the problem only for the case $v_a \geq v_b$, and the mechanism in the other case is symmetric. Proposition 2 below establishes that this reformulation is without loss for the seller.

Denote $\delta = v_a - v_b$, and notice that $\delta \in \left[-\overline{\delta}, \overline{\delta}\right]$ where $\overline{\delta} = \overline{v} - \underline{v}$. Suppose the seller offers a mechanism which consists of a set of messages $M = \{0, 1, 2\} \times \left[0, \overline{\delta}\right]$, allocation rules $q_1 : \left[0, \overline{\delta}\right] \to \left[\frac{1}{2}, 1\right]$, $q_2 : \left[0, \overline{\delta}\right] \to [1, 2]$ and payment rules $t_j : \left[0, \overline{\delta}\right] \to \mathbb{R}$ for $j = 1, 2$.[3] The buyer chooses how many units he would like to get and announces $\delta \in \left[0, \overline{\delta}\right]$. If 0 and δ are chosen, then the buyer receives the null allocation and there is no payment. If 1 and δ are chosen, then the buyer is assigned expected quantity $q_1(\delta)$ of the more preferred good, $1 - q_1(\delta)$ of the less preferred good, and pays $t_1(\delta)$ to the seller. If 2 and δ are chosen, then the buyer is assigned expected quantity $q_2(\delta)$ of the more preferred good, $2 - q_2(\delta)$ of the less preferred good, and pays $t_2(\delta)$ to the seller.

Let $u_j(\delta) := \delta q_j(\delta) - t_j(\delta)$ for every $\delta \in \left[0, \overline{\delta}\right]$, $j = 1, 2$. Consider a buyer with valuation (v_a, v_b) such that $v_a \geq v_b$, and let $\delta = v_a - v_b$. The utility of this buyer from participating in the seller's mechanism, and truthfully announcing his δ, is 0 if he chooses zero units, $u_1(\delta) + v_b$ if he chooses one unit, and $u_2(\delta) + 2v_b$ if he chooses two units. Thus the types such that $u_2(\delta) + v_b > u_1(\delta)$ and $u_2(\delta) + 2v_b > 0$ will buy two units. The types such that $u_1(\delta) > u_2(\delta) + v_b$ and $u_1(\delta) + v_b > 0$ will buy one unit. The types such that $u_2(\delta) + 2v_b < 0$ and $u_1(\delta) + v_b < 0$ will buy zero units. The types such that either $u_2(\delta) + v_b = u_1(\delta)$ or $u_1(\delta) + v_b = 0$ are indifferent between buying 1 and 2 units, or 0 and 1 units, but their choices will not affect the expected profit because these buyers' types have measure zero. Denote by $g_j(u_1(\delta), u_2(\delta), \delta)$ the measure of types with difference in the valuations δ who buy j units.

[3] Symmetry of the mechanism implies that buyers with $\delta = v_a - v_b \geq 0$ must get larger expected quantities of their preferred goods: $q_1(\delta) \geq 1 - q_1(\delta)$ and $q_2(\delta) \geq 2 - q_2(\delta)$. Otherwise it would be profitable for buyer with δ to report $-\delta$. The above constraints can be rewritten as $q_1(\delta) \geq 0.5$ and $q_2(\delta) \geq 1$.

Now let us state the seller's problem using this new notation. Note that IC constraints are rewritten using the envelope formula and monotonicity of the allocation.

$$\textbf{Program II}: \qquad \max_{q_j, u_j} \int_0^{\overline{\delta}} \left(\sum_{j=1,2} \left(\delta q_j(\delta) - u_j(\delta) - \sum_{k=1}^{j} c_k \right) g_j(u_1(\delta), u_2(\delta), \delta) \right) d\delta$$

subject to

F: $q_1(\delta) \in [0.5, 1]$, $q_2(\delta) \in [1, 2]$ for every $\delta \in \left[0, \overline{\delta}\right]$

IC: q_j nondecreasing, $u_j(\delta) = u_j(0) + \int_0^{\delta} q_j\left(\widetilde{\delta}\right) d\widetilde{\delta} \;\; \forall \delta \in \left[0, \overline{\delta}\right]$, $j = 1, 2$

Proposition 2 *Suppose mechanism* (q_1, q_2, u_1, u_2) *solves Program II. Then there exists an outcome equivalent mechanism* $(p_{a1}, p_{b1}, p_{a2}, p_{b2}, T)$ *that solves Program I.*

3.2 Properties of the solution and an example

Let us first consider some general properties of the optimal allocation.

Proposition 3 *It is without loss for the seller to consider mechanisms such that*

(i) For every δ, types with sufficiently high v_b buy two units.

(ii) Types with sufficiently high v_b and $|\delta|$ buy two units which are customized into their preferred product with probability 1.

(iii) Types with sufficiently low v_b and $|\delta|$ buy nothing.

(iv) Suppose for a given δ there are types that buy nothing, then there exists a nonempty set of values v_b that buy one unit.

Properties (i) and (ii) can be interpreted as *no distortion at the top* kind of results. Under Condition (1) it is efficient to sell two units of the preferred product to each type of the buyer. Property (i) says that high enough types will be served efficient quantity, although the allocation may still be inefficient because they may get a lottery over different products. Property (ii) says that there is a subset of high types that will be given a fully efficient allocation. Property (iii) is an *exclusion* result that is standard for multidimensional models.[4] Property (iv) shows that there is a subset of intermediate types that will be allocated one unit. This property holds because it costs less to produce the first unit than the second unit.

One implication of the result is that g_1 and g_2 (the measures of types who buy one and two units for a given difference in the valuations that appear in the objective of Program II) can be written in a relatively simple way.[5]

$$g_1(u_1(\delta), u_2(\delta), \delta) = \begin{cases} \int_{-u_1(\delta)}^{u_1(\delta)-u_2(\delta)} f(v_b+\delta, v_b)\, dv_b & \text{if} \quad \delta \in [0, \delta_1] \\ \int_{\underline{v}}^{u_1(\delta)-u_2(\delta)} f(v_b+\delta, v_b)\, dv_b & \text{if} \quad \delta \in [\delta_1, \delta_2] \\ 0 & \text{if} \quad \delta \in [\delta_2, \overline{\delta}] \end{cases} \tag{2}$$

and

$$g_2(u_1(\delta), u_2(\delta), \delta) = \begin{cases} \int_{u_1(\delta)-u_2(\delta)}^{\overline{v}-\delta} f(v_b+\delta, v_b)\, dv_b & \text{if} \quad \delta \in [0, \delta_2] \\ \int_{\underline{v}}^{\overline{v}-\delta} f(v_b+\delta, v_b)\, dv_b & \text{if} \quad \delta \in [\delta_2, \overline{\delta}] \end{cases} \tag{3}$$

where δ_1, δ_2 such that $0 \leq \delta_1 \leq \delta_2 \leq \overline{\delta}$, $u_1(\delta_1) + \underline{v} = 0$, and $u_2(\delta_2) - u_1(\delta_2) + \underline{v} = 0$.

In the Appendix we formulate the seller's problem as an optimal control problem and provide conditions for optimality. This approach can be applied to find a solution that is interior in the following sense: $\overline{v} - \delta > u_1(\delta) - u_2(\delta)$ on $[0, \delta_2]$ and $u_1(\delta) - u_2(\delta) > -u_1(\delta)$ on $[0, \delta_1]$. These conditions guarantee that the integrals in (2) and (3) do not vanish. Thus functions g_1, g_2, and the objective, are differentiable in u_1, u_2 (except for δ_1, δ_2) as required for applying the optimal control approach.[6]

[4] See, for example, Armstrong (1996).

[5] This is shown in Lemma 6 in the Appendix.

[6] All numerical examples with discretized type space that were computed appear to have interior solutions in the sense mentioned above. There is also a strong intuition that the optimal solution must be interior for every δ. First, two-unit contracts are more profitable than one-unit contracts, and thus not selling any such contracts should not be optimal. Second, if only a two-unit contract and a null contract are sold with positive probability, then offering a one-unit contract at half a price of the two-unit contract minus a small ε should be profitable because (a) it will attract roughly equal measures of buyers that were previously buying nothing and that were buying two units, (b) its cost is strictly less than half the cost of a two-unit contract. However, formalizing this intuition appears to be very challenging.

The optimality conditions reveal how the profit is affected by an increase in $q_k(\delta)$ for a given $\delta \in [0, \overline{\delta}]$ and $k \in \{1, 2\}$:[7]

$$\underbrace{\delta g_k (u_1(\delta), u_2(\delta), \delta) - \int_{\delta}^{\overline{\delta}} g_k \left(u_1\left(\widetilde{\delta}\right), u_2\left(\widetilde{\delta}\right), \widetilde{\delta}\right) d\widetilde{\delta}}_{\text{rent extraction effect}}$$

$$\underbrace{+\sum_{j=1,2} \int_{\delta}^{\overline{\delta}} \frac{\partial g_j \left(u_1\left(\widetilde{\delta}\right), u_2\left(\widetilde{\delta}\right), \widetilde{\delta}\right)}{\partial u_k} \left(\widetilde{\delta} q_j\left(\widetilde{\delta}\right) - u_j\left(\widetilde{\delta}\right) - \sum_{m=1}^{j} c_m\right) d\widetilde{\delta}}_{\text{participation effects}}$$

The first effect can be called a *rent extraction effect*, and it is standard for mechanism design problems with asymmetric information. Raising $q_k(\delta)$ means that the buyers with difference in the valuations equal to δ who choose to buy k units are more likely to get their preferred product. This increases the surplus by δ, and the measure of these buyers is given by $g_k(u_1(\delta), u_2(\delta), \delta)$. However raising $q_k(\delta)$ also increases informational rent u_k for all buyers with differences in the valuations above δ who choose to buy k units. The expected loss from these types of buyers is $\int_{\delta}^{\overline{\delta}} g_k \left(u_1\left(\widetilde{\delta}\right), u_2\left(\widetilde{\delta}\right), \widetilde{\delta}\right) d\widetilde{\delta}$.

The second group of effects accounts for the changes on the extensive margin, and these can be called *participation effects*. An increase in the informational rent, $u_k\left(\widetilde{\delta}\right)$ for $\widetilde{\delta} > \delta$, attracts additional buyers with difference in the valuations equal to $\widetilde{\delta}$ to buy k units, and change the measure of such buyers by $\frac{\partial}{\partial u_k} g_k \left(u_1\left(\widetilde{\delta}\right), u_2\left(\widetilde{\delta}\right), \widetilde{\delta}\right) \geq 0$. This change has to be multiplied by the profit from this category of buyers, $\widetilde{\delta} q_k\left(\widetilde{\delta}\right) - u_k\left(\widetilde{\delta}\right) - \sum_{m=1}^{k} c_m$. But an increase in $u_k\left(\widetilde{\delta}\right)$ may attract some buyers with difference in the valuations equal to $\widetilde{\delta}$ away from buying $j \neq k$ units, and change the measure of such buyers by $\frac{\partial}{\partial u_k} g_j \left(u_1\left(\widetilde{\delta}\right), u_2\left(\widetilde{\delta}\right), \widetilde{\delta}\right) \leq 0$. This change has to be multiplied by the respective profit, $\widetilde{\delta} q_j\left(\widetilde{\delta}\right) - u_j\left(\widetilde{\delta}\right) - \sum_{m=1}^{j} c_m$.

These participation effects work differently for the case of one unit and two units. An increase in $q_2(\delta)$ may attract additional buyers into purchasing two units only at the expense of the buyers who are currently purchasing one unit: $\frac{\partial g_2}{\partial u_2} = -\frac{\partial g_1}{\partial u_2} \geq 0$. The sum of partici-

[7] This expression is equivalent to (16) in the Appendix.

pation effects when $k = 2$ is nonnegative because the profit from two units is not less than from one unit. On the other hand an increase in $q_1(\delta)$ usually attracts additional buyers into purchasing one unit from two sources: buyers switching from purchasing two units and buyers switching from purchasing nothing. The effect of attracting the former buyers is non-positive since the profit from one unit not higher than profit from two units, but the effect of attracting the new buyers on profit is positive.

Next we consider an example.

Example 1 *Let $c_1 = 0$, $c_2 = c > 0$, and the valuations be uniformly distributed on $[c, c+1]^2$. The optimal mechanism offers the buyer the following options:*[8]

(i) When $c \in (0, 1]$:

$$\begin{cases} (0,0) \ \ \textit{at a price } 0 \\ (\alpha(c), 1-\alpha(c)), (1-\alpha(c), \alpha(c)) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+3\alpha(c)} + \frac{2}{3}c \\ (2,0), (0,2) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+3\alpha(c)} + \frac{5}{3}c + \frac{1}{3}\sqrt{3(2-\alpha(c))} \end{cases}$$

where α is decreasing in c, $\alpha(0) = 1$, $\alpha(1) = \frac{2}{3}$.[9]

(ii) When $c \in (1, c^*)$ (where $c^* \approx 1.22$):

$$\begin{cases} (0,0) \ \ \textit{at a price } 0 \\ (\widetilde{\alpha}(c), 1-\widetilde{\alpha}(c)), (1-\widetilde{\alpha}(c), \widetilde{\alpha}(c)) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+3\widetilde{\alpha}(c)} + \frac{2}{3}c \\ (1+\widetilde{\alpha}(c), 1-\widetilde{\alpha}(c)), (1-\widetilde{\alpha}(c), 1+\widetilde{\alpha}(c)) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+3\widetilde{\alpha}(c)} + \frac{5}{3}c + \frac{2}{3} \\ (2,0), (0,2) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+3\widetilde{\alpha}(c)} + \frac{5}{3}c + \frac{2}{3} + (1-\widetilde{\alpha}(c))\left(\frac{2}{3} - \frac{1}{3}\sqrt{\frac{2-\widetilde{\alpha}(c)}{1-\widetilde{\alpha}(c)}}\right) \end{cases}$$

where $\widetilde{\alpha}$ is decreasing in c, $\widetilde{\alpha}(1) = \frac{2}{3}$, $\widetilde{\alpha}(c^*) = \frac{1}{2}$.[10]

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[8] Each option specifies expected quantitites of product a and b, and a price.

[9] $\alpha(c)$ solves $\left(\sqrt{c^2+3\alpha}+c\right)^2 = \sqrt{3(2-\alpha)}\left(\sqrt{c^2+3\alpha}+2c\right)$.

[10] $\widetilde{\alpha}(c)$ solves $\left(\sqrt{c^2+3\alpha}+c\right)^2\left(2+\sqrt{\frac{2-\alpha}{1-\alpha}}+\sqrt{\frac{1-\alpha}{2-\alpha}}\right) = 9\left(\sqrt{c^2+3\alpha}+2c\right)$.

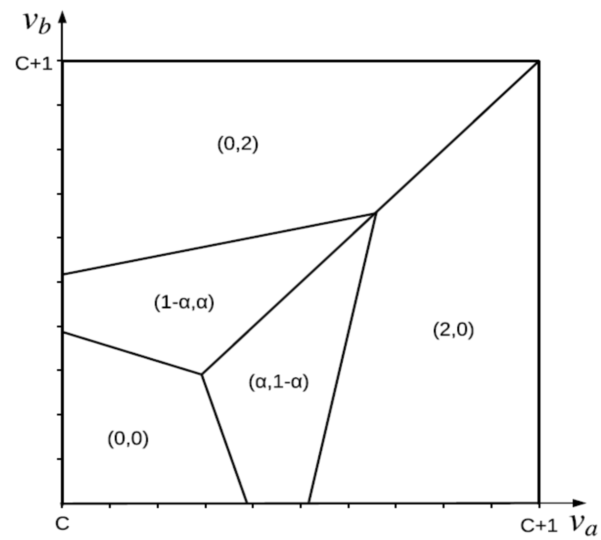

Figure 1: Optimal allocation in Example 1 when $c \in (0,1]$.

(iii) When $c \in [c^*, \infty)$:

$$\begin{cases} (0,0) \ \ \textit{at a price } 0 \\ \left(\frac{1}{2},\frac{1}{2}\right) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+\frac{3}{2}}+\frac{2}{3}c \\ \left(\frac{3}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{3}{2}\right) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+\frac{3}{2}}+\frac{5}{3}c+\frac{2}{3} \\ (2,0),(0,2) \ \ \textit{at a price } \frac{1}{3}\sqrt{c^2+\frac{3}{2}}+\frac{5}{3}c+1-\frac{1}{6}\sqrt{3} \end{cases}$$

One notable feature of this example is that the optimal mechanism is quite simple, i.e. contains only a few point contracts. For a model with a single unit there are results that show optimality of simple mechanisms under certain distributional assumptions (Wang and Tang, 2017; Thirumulanathan et al, 2019a, 2019b). Example 1 suggests that this property is likely to hold for models with more than one unit.

Note that the offered two-unit contracts are always deterministic and efficient when $c \in (0,1]$, and are deterministic and efficient for most of the buyers' types when $c > 1$. On the other hand the offered one-unit contracts are always stochastic. Thus the seller introduces two distortions for the buyers whose valuations are at the intermediate level. First, they are served only one unit while it would be efficient to give them two units. Second, they

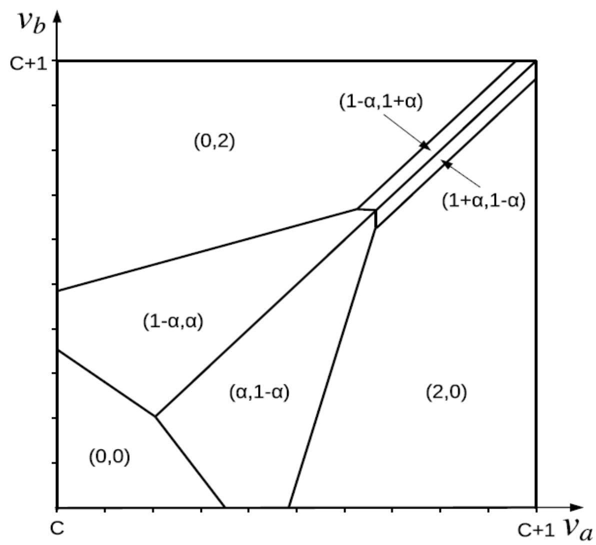

Figure 2: Optimal allocation in Example 1 when $c \in (1, c^*)$.

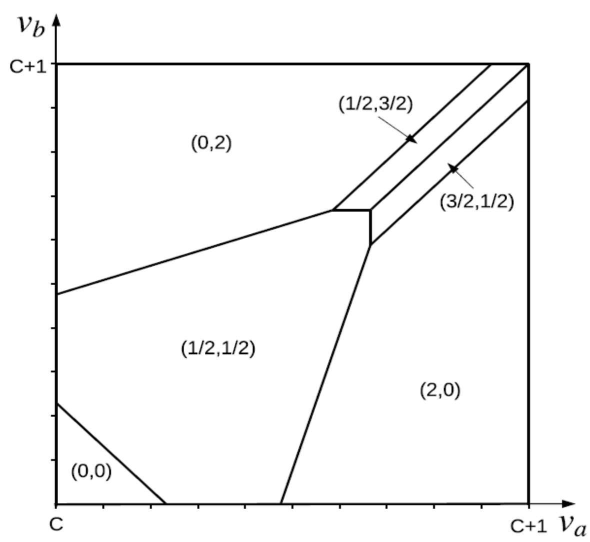

Figure 3: Optimal allocation in Example 1 when $c \in [c^*, 1)$.

get a lottery over products rather than their preferred product. In the next subsection we compare the optimal mechanism for selling two units with the optimal mechanisms for selling each unit of the good separately to obtain additional insights into the form of the optimal mechanism.

3.3 Comparison with selling units separately

A single-product multiple-unit scenario when the unobserved buyers' tastes are described by a one-dimensional parameter had been extensively studied.[11] One interesting property of the solution to that class of problems when the seller's marginal costs are increasing is that it can be obtained by treating the sale of each individual unit as if it was done on a separate market, regardless of the other units.[12] In this subsection we investigate under what circumstances a similar property holds for our model.

Let us first revisit the setting of Example 1 and consider the optimal mechanisms for separate selling of units 1 and 2. Solutions are translated from Pavlov (2011b) to the present setting.

Example 2 *Let $c_1 = 0$, $c_2 = c > 0$, and the valuations be uniformly distributed on $[c, c+1]^2$. The optimal mechanism for selling the first unit is to offer the buyer the following options:*

(i) When $c \in (0, 1]$:

$$\begin{cases} (0,0) & \text{at a price } 0 \\ (1,0),(0,1) & \text{at a price } \frac{2}{3}c + \frac{1}{3}\sqrt{c^2+3} \end{cases}$$

[11] Mussa and Rosen (1978) is a classic reference. For a recent general approach to the problem see Hellwig (2010).

[12] See, for example, Section 2.2 in Armstrong, (2016) for description of this "demand profile" approach. Suppose the gross surplus of the buyer with valuation $v \in \mathbb{R}_+$ who consumes quantity $q \in \mathbb{R}_+$ is $vw(q)$ where w is an increasing (weakly) concave function with $w(0) = 0$. Then the optimal marginal price for qth unit also solves the problem of optimally selling only this unit provided that the demand is computed using the marginal value $vw'(q)$.

(ii) When $c \in (1, c^{**})$ *(where* $c^{**} \approx 1.372$*):*

$$\begin{cases} (0,0) \;\; \textit{at a price } 0 \\ (\beta(c), 2-\beta(c)), (1-\beta(c), \beta(c)) \;\; \textit{at a price } \frac{1}{3}c + \frac{3}{8} + \frac{1}{8}\sqrt{16c+9} \\ (1,0), (0,1) \;\; \textit{at a price } \frac{41}{96} + \frac{1}{3}c + \left(\frac{1}{12}c + \frac{1}{32}\right)\sqrt{16c+9} \end{cases}$$

where $\beta(c) = \frac{27}{16} + \left(\frac{9}{16} - \frac{1}{2}c\right)\sqrt{16c+9}$.

(iii) When $c \in [c^{**}, \infty)$*:*

$$\begin{cases} (0,0) \;\; \textit{at a price } 0 \\ (0.5, 0.5) \;\; \textit{at a price } \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + \frac{3}{2}} \\ (1,0), (0,1) \;\; \textit{at a price } \frac{1}{6} + \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + \frac{3}{2}} \end{cases}$$

The optimal mechanism for selling the second unit is to offer the buyer the following options:

$$\begin{cases} (0,0) \;\; \textit{at a price } 0 \\ (1,0), (0,1) \;\; \textit{at a price } \frac{1}{3}\sqrt{3} + c \end{cases}$$

Comparing Examples 1 and 2 we can see that the solutions are always different.[13,14] Both solutions allow buying two units of any particular product, $(2,0), (0,2)$, but the price in the optimal mechanism for joint selling is lower than in the solution for separate selling, except for $c = 1$ when the prices are equal. Next, the separate selling solution always gives an option to purchase deterministic one-unit contracts, $(1,0), (0,1)$, but those are never offered in the joint selling mechanism (or we can think that those options are offered at prohibitively high prices). When $c \in (0,1]$, the solution for separate selling is deterministic, while the optimal mechanism for joint selling offers stochastic contracts in case one unit is sold. When $c \in (1, c^{**}]$, both solutions offer one-unit and two-unit stochastic contracts, but they differ

[13] Solutions coincide when $c = 0$, however in this the unit costs are not strictly increasing.

[14] As mentioned earlier in this section, solutions will coincide when there is a single product. For example, let $c_1 = 0$, $c_2 = c > 0$, and the valuation is uniformly distributed on $[c, c+1]$. Then the optimal mechanism offers to sell the first unit at $\max\left\{\frac{1}{2} + \frac{1}{2}c, c\right\}$, and the second unit at $\frac{1}{2} + c$.

in prices and probabilities of getting different products. When $c > c^{**}$, both solutions offer the same stochastic one-unit and two-unit contracts, $(0.5, 0.5)$ is priced in the same way, but $\left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right)$ are priced differently.

Let us take a closer look at the optimal joint selling and separate selling mechanisms when $c = 1$ and $c = 2$. The optimal separate selling mechanism when $c = 1$ offers for purchase deterministic one-unit contracts and two-unit contracts. The optimal joint selling mechanism simply replaces deterministic one-unit contracts with lotteries $\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ and lowers the price by about 6.7%. The profit gain from joint selling relative to separate selling is about 0.46%.

The optimal separate selling mechanism when $c = 2$ offers for purchase a one-unit lottery $\left(\frac{1}{2}, \frac{1}{2}\right)$, one-unit deterministic contracts, two-unit lotteries $\left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right)$, and two-unit deterministic contracts. The optimal joint selling mechanism removes the one-unit deterministic contracts (or raises their prices to a prohibitively high level), and adjust the prices of two-unit lotteries and deterministic contracts up by about 1.9% and down by about 0.7%, respectively. The profit gain from joint selling relative to separate selling is about 0.5%.[15]

Why cannot the separate selling mechanism replicate the joint selling optimal mechanism? Let us revisit again the case when $c = 1$, and consider conditions on the prices in the individual selling mechanisms that have to hold if it were to replicate the outcome of the optimal joint selling mechanism. Since some buyers' types are supposed to receive two units of their preferred good, it must be that the separate selling mechanisms offer options to buy deterministic contracts for unit 1 and 2. Clearly the price of deterministic contracts for unit 1 must be greater than for the lotteries $\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$, otherwise no one would buy the lotteries. However then the buyers' types sufficiently close to the diagonal, $v_a \approx v_b$, would buy lottery for the first unit and deterministic contract for unit 2 rather than deterministic contracts for both units.

[15]The profit gain from joint selling relative to separate selling is under 1% for every c. While this is not a large gain, it is interesting to note that it is of the same magnitude as the gain from using stochastic mechanisms as opposed to deterministic ones in a one-unit model in a similar example (Pavlov, 2011b).

On the other hand, notice that separate selling mechanism can replicate the optimal joint selling mechanism if we allow the prices in the mechanism for selling the second unit to depend on the contract chosen for unit 1. In case $c = 1$, deterministic contract for unit 2 can be priced at prohibitively high price if the buyer purchased a lottery for unit 1.

The next result gives sufficient conditions for the separate selling approach and the fully optimal mechanism design to result in the same solution.

Proposition 4 *Suppose the fully optimal joint selling solution is deterministic and conditionally efficient, i.e. it only offers options* $(0,0)$*,* $(1,0)$*,* $(0,1)$*,* $(2,0)$*, and* $(0,2)$*. Then the solution to the separate selling problem coincides with the fully optimal joint selling solution.*

Next we provide an example with such a property. This example does not satisfy condition (1) in Proposition 1, and thus it cannot be solved by the technique used earlier. Instead it is solved by "integration along rays" (Armstrong, 1996). The idea is to find an optimal mechanism for the types that lie on a given ray from the origin, and then check if the combination of all mechanisms is globally incentive compatible on the whole type space.

Example 3 *Let* $0 \le c_1 \le c_2 \le 1$*, and the valuations be uniformly distributed on* $[0,1]^2$*. The optimal mechanism offers the buyer the following options:*

$$\begin{cases} (0,0) \ \ \textit{at a price } 0 \\ (1,0)\,,(0,1) \ \ \textit{at a price } \frac{1}{3}c_1 + \frac{1}{3}\sqrt{c_1^2+3} \\ (2,0)\,,(0,2) \ \ \textit{at a price } \frac{1}{3}c_1 + \frac{1}{3}\sqrt{c_1^2+3} + \frac{1}{3}c_2 + \frac{1}{3}\sqrt{c_2^2+3} \end{cases}$$

4 Nondecreasing returns

Now let us consider the case when unit 1 is at least as expensive as unit 2: $c_1 \ge c_2 \ge 0$. If $c_1 > c_2$, then the seller will take advantage of the economies of scale as follows. The optimal way to organize production of expected quantity $Q \in [0,2]$ is to produce two units with probability $\frac{1}{2}Q$ and zero units with probability $1 - \frac{1}{2}Q$. Thus the cost of producing Q

units is $\frac{1}{2}(c_1+c_2)Q$. In the context of Program I this means that the third set of feasibility constraints will be binding, and thus $\sum_{i=a,b} p_{i1}(v) = \sum_{i=a,b} p_{i2}(v)$ for every v. If $c_1 = c_2$, then there are multiple solutions to the seller's problem, including the ones where the constraints mentioned in the previous sentence bind as well.

Moreover notice the buyer cares only about the expected quantity of each product he receives. Thus there is no loss for the seller in restricting attention to mechanisms such that $p_{i1}(v) = p_{i2}(v)$ for $i = a, b$ for every v. Let us state it as a result and omit the proof.

Lemma 2 *Let $c_1 \geq c_2$. Without loss for every v we can set $p_{i1}(v) = p_{i2}(v)$ for $i = a, b$.*

Denote $\overline{p}_i(v) = p_{ij}(v)$ for $i = a, b$, $j = a, b$. Then seller's problem can be rewritten as follows.

$$\textbf{Program III}: \qquad \max_{\overline{p}_i, T} E\left[T(v) - (c_1+c_2)\left(\sum_{i=a,b} \overline{p}_i(v)\right)\right] \qquad \text{subject to}$$

F: $\overline{p}_i(v) \geq 0$ for every $v \in \Theta$, $i = a, b$

$$\sum_{i=a,b} \overline{p}_i(v) \leq 1 \text{ for every } v \in \Theta$$

IC: $\sum_{i=a,b} 2v_i\overline{p}_i(v) - T(v) \geq \sum_{i=a,b} 2v_i\overline{p}_i(v') - T(v')$ for every $v, v' \in \Theta$

IR: $\sum_{i=a,b} 2v_i\overline{p}_i(v) - T(v) \geq 0$ for every $v \in \Theta$

If we divide the objective, IC and IR constraints by 2, then it becomes clear that this problem is isomorphic to a problem where the seller has a single unit of a good that can be produced at a cost $\frac{1}{2}(c_1+c_2)$. This problem was studied in Pavlov (2011b), and we can just translate some of the results into the present setting.

Proposition 5 *Let $c_1 \geq c_2$ and suppose*

$$3f(v) + \sum_{i=a,b}\left(v_i - \frac{c_1+c_2}{2}\right)\frac{\partial f(v)}{\partial v_i} \geq 0 \textit{ and } \frac{c_1+c_2}{2} \leq \underline{v}.$$

Then there is no loss for the seller in optimizing over mechanisms that satisfy

$$\sum_{i=a,b} \overline{p}_i(v) \in \{0,1\} \text{ for every } v \in \Theta$$

An approach similar to the one used in Section 3 can be used to study the optimal contract. The general properties of the solution stated in parts (i), (ii), and (iii) of Proposition 3 remain valid in this setting. However, property (iv) of that result does not hold here: it is never optimal to offer for sale just one unit of the good. The following example is translated from Pavlov (2011b) to the present setting.

Example 4 *Denote* $c := \frac{1}{2}(c_1 + c_2)$*, and let the valuations be uniformly distributed on* $[c+d, c+d+1]^2$ *where* $c, d \geq 0$*. The optimal mechanism offers the buyer the following options:*

(i) When $d \in [0,1]$*:*

$$\begin{cases} (0,0) & \text{at a price } 0 \\ (2,0),(0,2) & \text{at a price } \frac{4}{3}d + \frac{2}{3}\sqrt{d^2+3} + 2c \end{cases}$$

(ii) When $d \in (1, d^*)$ *(where* $d^* \approx 1.372$*):*

$$\begin{cases} (0,0) & \text{at a price } 0 \\ (\beta(d), 2-\beta(d)), (2-\beta(d), \beta(d)) & \text{at a price } \frac{2}{3}d + \frac{3}{4} + \frac{1}{4}\sqrt{16d+9} + 2c \\ (2,0),(0,2) & \text{at a price } \frac{41}{48} + \frac{2}{3}d + \left(\frac{1}{6}d + \frac{1}{16}\right)\sqrt{16d+9} + 2c \end{cases}$$

where $\beta(d) = \frac{27}{16} + \left(\frac{9}{16} - \frac{1}{2}d\right)\sqrt{16d+9}$*.*

(iii) When $d \in [d^*, \infty)$*:*

$$\begin{cases} (0,0) & \text{at a price } 0 \\ (1,1) & \text{at a price } \frac{4}{3}d + \frac{2}{3}\sqrt{d^2+\frac{3}{2}} + 2c \\ (2,0),(0,2) & \text{at a price } \frac{1}{3} + \frac{4}{3}d + \frac{2}{3}\sqrt{d^2+\frac{3}{2}} + 2c \end{cases}$$

5 Appendix

5.1 Proofs for Section 3.1

We will prove several preliminary Lemmas before proving Proposition 1.

Lemma 3 *In the optimal mechanism* $U(\underline{v},\underline{v})=0$. *There is no loss for the seller in setting* $p_{ij}(\underline{v},\underline{v})=0$ *for every* i,j *and* $T(\underline{v},\underline{v})=0$.

Proof. Note that for every $v\in\Theta$ IC implies

$$U(v)\geq U(\underline{v},\underline{v})+\sum_{i=a,b}\sum_{j=1,2}(v_i-\underline{v})\,p_{ij}(\underline{v},\underline{v})\geq U(\underline{v},\underline{v})$$

If $U(\underline{v},\underline{v})>0$, then we can increase profits by raising all payments T by $U(\underline{v},\underline{v})$. Thus it is optimal to set $U(\underline{v},\underline{v})=0$.

Assigning a null allocation for free to the lowest type achieves $U(\underline{v},\underline{v})=0$ and does not affect expected profit because the lowest type has measure zero. ■

Lemma 4 *Let* $v=(v_a,v_b)$, $v'=(v'_a,v'_b)$ *such that* $v_a-v_b=v'_a-v'_b$ *and* $v_b>v'_b$. *Suppose* v *prefers allocation* $(p_{a1},p_{b1},p_{a2},p_{b2})$ *at price* T *to allocation* $(p'_{a1},p'_{b1},p'_{a2},p'_{b2})$ *at price* T', *and* v' *has the opposite preference. Then IC implies* $\sum_{i=a,b}\sum_{j=1,2}p_{ij}\geq\sum_{i=a,b}\sum_{j=1,2}p'_{ij}$.

Proof. Denote $\delta=v_a-v_b$. We are given that

$$\delta\left(\sum_{j=1,2}p_{aj}\right)+v_b\left(\sum_{i=a,b}\sum_{j=1,2}p_{ij}\right)-T\geq\delta\left(\sum_{j=1,2}p'_{aj}\right)+v_b\left(\sum_{i=a,b}\sum_{j=1,2}p'_{ij}\right)-T'$$
$$\delta\left(\sum_{j=1,2}p'_{aj}\right)+v'_b\left(\sum_{i=a,b}\sum_{j=1,2}p'_{ij}\right)-T'\geq\delta\left(\sum_{j=1,2}p_{aj}\right)+v'_b\left(\sum_{i=a,b}\sum_{j=1,2}p_{ij}\right)-T$$

Sum up the inequalities and simplify to get

$$(v_b-v'_b)\left(\sum_{i=a,b}\sum_{j=1,2}p_{ij}-\sum_{i=a,b}\sum_{j=1,2}p'_{ij}\right)\geq 0$$

■

Lemma 5 *Let* $\Delta c = c_2 - c_1$*, and* $\Theta_1 = \{v \in \Theta : p_{a1}(v) + p_{b1}(v) = 1\}$*. Expected profit can be expressed as follows*

$$\sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} U(\overline{v}, v_{-i})(\overline{v} - c_1) f(\overline{v}, v_{-i}) \, dv_{-i} - \sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} U(\underline{v}, v_{-i})(\underline{v} - c_1) f(\underline{v}, v_{-i}) \, dv_{-i}$$
$$- \int_{\Theta} U(v) \left(3f(v) + \sum_{i=a,b} (v_i - c_1) \frac{\partial f(v)}{\partial v_i} \right) dv$$
$$+ \Delta c \int_{\partial \Theta_1} \left(U(v) - \frac{v_a + v_b}{2} \right) \mathbf{n}(v) f(v) \, dv - \Delta c \int_{\Theta_1} \left(U(v) - \frac{v_a + v_b}{2} \right) \left(\sum_{i=a,b} \frac{\partial f(v)}{\partial v_i} \right) dv$$

where $\partial \Theta_1$ *is the surface of set* Θ_1*, and* $\mathbf{n}(v)$ *is the outward-pointing unit normal vector at* v *on the surface of* Θ_1*.*

Proof. The expected profit is

$$\begin{aligned} & E\left[T(v) - \sum_{i=a,b} \sum_{j=1,2} c_j p_{ij}(v) \right] \\ = \; & E\left[\sum_{i=a,b} (v_i - c_1) \left(\sum_{j=1,2} p_{ij}(v) \right) - U(v) \right] - E\left[\Delta c \sum_{i=a,b} p_{i2}(v) \right] \end{aligned}$$

By the envelope theorem we know that $\frac{\partial U(v)}{\partial v_i} = \sum_{j=1,2} p_{ij}(v)$ a.e. for $i = a, b$. Using this fact and the divergence theorem[16] the first term in the expression for the profit above can be written as

$$\sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} U(\overline{v}, v_{-i})(\overline{v} - c_1) f(\overline{v}, v_{-i}) \, dv_{-i} - \sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} U(\underline{v}, v_{-i})(\underline{v} - c_1) f(\underline{v}, v_{-i}) \, dv_{-i}$$
$$- \int_{\Theta} U(v) \left(3f(v) + \sum_{i=a,b} (v_i - c_1) \frac{\partial f(v)}{\partial v_i} \right) dv$$

[16] See, for example, Armstrong (1996).

and the subtracted second term can be written as

$$\begin{aligned}
&\Delta c \int_{\Theta_1} \left(\sum_{i=a,b} \frac{\partial U(v)}{\partial v_i} - 1 \right) f(v)\, dv && (4) \\
= \ &\Delta c \int_{\partial\Theta_1} \left(U(v) - \frac{v_a + v_b}{2} \right) \mathbf{n}(v) f(v)\, dv - \Delta c \int_{\Theta_1} \left(U(v) - \frac{v_a + v_b}{2} \right) \left(\sum_{i=a,b} \frac{\partial f(v)}{\partial v_i} \right) dv
\end{aligned}$$

■

Proof of Proposition 1. Consider an admissible mechanism $(p_{a1}, p_{b1}, p_{a2}, p_{b2}, T)$ that generates utility schedule U, and suppose that $\sum_{i=a,b} \sum_{j=1,2} p_{ij}(v) \notin \{0, 1, 2\}$ for some types. We will make three adjustments to the original mechanism, each adjustment will produce an admissible mechanism and weakly increase profits. The first adjustment will ensure that the types on the upper boundary receive 2 units in total. The second adjustment will ensure that some specific types receive 1 unit in total. The last adjustment will remove all contracts other than those that give exactly 0, 1, or 2 units in total.

Step 1. Denote

$$\overline{\partial\Theta} = \{v \in \Theta \mid \text{there is } h \in \{a, b\} \text{ such that } v_h = \overline{v}\}$$

For every $v \in \overline{\partial\Theta}$ modify the allocation and payment as follows. If $v_a = \overline{v}$, then $\widehat{p}_{bj}(v) = p_{bj}(v)$, $\widehat{p}_{aj}(v) = 1 - p_{bj}(v)$ for $j = 1, 2$. If $v_b = \overline{v}$ and $v_a \neq \overline{v}$, then $\widehat{p}_{aj}(v) = p_{aj}(v)$, $\widehat{p}_{bj}(v) = 1 - p_{aj}(v)$ for $j = 1, 2$. Also let

$$\widehat{T}(v) = T(v) + \overline{v} \left(2 - \sum_{i=a,b} \sum_{j=1,2} p_{ij}(v) \right)$$

All types that do not belong to $\overline{\partial\Theta}$ get the same allocation and payment as before.

Note that this adjustment does not affect utility of types in $\overline{\partial\Theta}$. It does not disturb IC because for every $v \in \Theta$ and every $\widetilde{v} \in \overline{\partial\Theta}$, with $\widetilde{v}_i = \overline{v}$:

$$
\begin{aligned}
& v_a \left(\sum_{j=1,2} \widehat{p}_{aj}(\widetilde{v}) \right) + v_b \left(\sum_{j=1,2} \widehat{p}_{bj}(\widetilde{v}) \right) - \widehat{T}(\widetilde{v}) \\
= \; & v_a \left(\sum_{j=1,2} p_{aj}(\widetilde{v}) \right) + v_b \left(\sum_{j=1,2} p_{bj}(\widetilde{v}) \right) - T(\widetilde{v}) - (\overline{v} - v_i) \left(2 - \sum_{i=a,b} \sum_{j=1,2} p_{ij}(v) \right) \\
\leq \; & v_a \left(\sum_{j=1,2} p_{aj}(\widetilde{v}) \right) + v_b \left(\sum_{j=1,2} p_{bj}(\widetilde{v}) \right) - T(\widetilde{v}) \\
\leq \; & v_a \left(\sum_{j=1,2} p_{aj}(v) \right) + v_b \left(\sum_{j=1,2} p_{bj}(v) \right) - T(v) = U(v)
\end{aligned}
$$

where the first equality is by definition of $\widehat{p}_{ij}$ and $\widehat{T}$, the second inequality is by IC of the original mechanism. The expected profits are not affected because $\overline{\partial\Theta}$ has measure zero.

Step 2. Denote by Σ the set of available expected allocations in the mechanism: all (Q_a, Q_b) such that $(Q_a, Q_b) = \left(\sum_{j=1,2} p_{aj}(v), \sum_{j=1,2} p_{bj}(v) \right)$ for some v. Note that $(0,0) \in \Sigma$ by Lemma 3. Define a pricing function over Σ as follows: $t(Q_a, Q_b) = T(v)$ if $(Q_a, Q_b) = \left(\sum_{j=1,2} p_{aj}(v), \sum_{j=1,2} p_{bj}(v) \right)$. If a given pair of quantities (Q_a, Q_b) can be achieved via different reports, then IC implies that the payment is the same for all such reports. Let $\overline{\Sigma}$ be the convex hull of closure of Σ, i.e. $\overline{\Sigma} = co(cl(\Sigma))$. Let $\overline{t}$ be the greatest convex function that is weakly below t, i.e. $\overline{t}(Q) = vex(t(Q))$ for every $Q \in \overline{\Sigma}$.

Suppose now that we allow the buyer to choose any contract from $(Q_a, Q_b) \in \overline{\Sigma}$ at a price $\overline{t}(Q_a, Q_b)$. Consider the optimal choice correspondence $\pi : \Theta \rightrightarrows \overline{\Sigma}$. It is upper hemicontinuous, nonempty, compact- and convex-valued (by the Maximum theorem and due to quasi-concavity of the objective). Note that IC of the original mechanism implies that the original contract assigned to each type of the buyer remains optimal: $\left(\sum_{j=1,2} p_{aj}(v), \sum_{j=1,2} p_{bj}(v) \right) \in \pi(v)$ for every v.[17]

[17] Suppose not, i.e. there exists $v \in \Theta$ such that $(Q_a^*, Q_b^*) \in \pi(v)$ and $v_a Q_a^* + v_b Q_b^* - \overline{t}(Q_a^*, Q_b^*) > U(v)$. Let $\tau(Q_a, Q_b) := \max\left\{ \overline{t}(Q_a, Q_b), v_a Q_a + v_b Q_b - U(v) \right\}$ for every $(Q_a, Q_b) \in \overline{\Sigma}$. Function τ is convex, lies weakly below t, and $\overline{t}(Q_a^*, Q_b^*) < \tau(Q_a^*, Q_b^*)$. This contradicts the definition of $\overline{t}$.

Define another correspondence $s : \Theta \rightrightarrows [0,2]$ as follows:

$$s(v) = \{s \in [0,2] : \exists (Q_a, Q_b) \in \pi(v) \text{ such that } s = Q_a + Q_b\}$$

Note that s is also upper hemicontinuous, nonempty, compact- and convex-valued.

Fix $\delta \in [\underline{v} - \overline{v}, \overline{v} - \underline{v}]$. Note that Lemma 4 implies that $s(v_b + \delta, v_b)$ is nondecreasing in v_b: if $v_b < v_b'$, then for every $\sigma \in s(v_b + \delta, v_b)$ and $\sigma' \in s(v_b' + \delta, v_b')$ we have $\sigma \leq \sigma'$. Adjustment in Step 1 ensured that $2 \in s(v_b + \delta, v_b)$ if $(v_b + \delta, v_b) \in \overline{\partial\Theta}$. Note that we can also choose $v_b \in \mathbb{R}$ low enough so that $0 \in s(v_b + \delta, v_b)$ (it may have to be so low that $(v_b + \delta, v_b) \notin \Theta$). Thus there exists non-empty $I_b(\delta) \subseteq \mathbb{R}$ such that $1 \in s(v_b + \delta, v_b)$ for every $v_b \in I_b(\delta)$. Suppose not. Then by monotonicity of s there exists v_b' such that $s(v_b + \delta, v_b) \subseteq [0, 1)$ for every $v_b < v_b'$, $s(v_b + \delta, v_b) \subseteq (1, 2]$ for every $v_b > v_b'$. Upper hemicontinuity implies that there exists $\sigma' < 1$ and $\sigma'' > 1$ such that $\sigma', \sigma'' \in s(v_b' + \delta, v_b')$. Convex-valuedness then implies $1 \in s(v_b' + \delta, v_b')$.

Next perform the following adjustment to the mechanism for every $v \in I_b(\delta)$ such that $v \notin \overline{\partial\Theta}$. Note that there exists some $(Q_a, Q_b) \in \pi(v)$ such that $Q_a + Q_b = 1$. Adjust the contract for v as follows: $\widehat{p}_{a1}(v) = Q_a$, $\widehat{p}_{b1}(v) = Q_b$, $\widehat{p}_{a2}(v) = \widehat{p}_{b2}(v) = 0$, $\widehat{T}(v) = \overline{t}(Q_a, Q_b)$.

Such a mechanism adjustment is IC and gives the same utility U to every type of the buyer as the original mechanism. By Lemma 5 the expected profit is completely determined by U and the set $\Theta_1 = \{v \in \Theta : p_{a1}(v) + p_{b1}(v) = 1\}$. The adjustment of the mechanism may have expanded set Θ_1, but the contribution of these new points to the expected profit is zero because for every such point v we have $\widehat{p}_{a2}(v) = \widehat{p}_{b2}(v) = 0$. Thus the expected profit remains the same.

Step 3. Construct a new menu of contracts by removing all contracts for which the expected quantity does not equal exactly to 0, 1 or 2:

$$\Sigma' = \left\{(Q_a, Q_b) \in \overline{\Sigma} : Q_a + Q_b \in \{0, 1, 2\}\right\}$$

Suppose now that we allow the buyer to choose any contract from $(Q_a, Q_b) \in \Sigma'$ at price $\bar{t}(Q_a, Q_b)$ determined in Step 2. Denote the utility from this new mechanism by $\widehat{U}$. Since we are just removing options from the original mechanism, we should have $\widehat{U}(v) \leq U(v)$ for every $v \in \Theta$.

Similarly to Step 2 consider the optimal choice correspondence $\widehat{\pi} : \Theta \rightrightarrows \Sigma'$, correspondence $\widehat{s} : \Theta \rightrightarrows \{0, 1, 2\}$, and sets $\widehat{I}_b(\delta)$ for every $\delta \in [\underline{v} - \overline{v}, \overline{v} - \underline{v}]$. Similarly to Step 2 make a mechanism adjustment so that all types in $\widehat{I}_b(\delta)$ are assigned some contract such that $\widehat{p}_{a1}(v) + \widehat{p}_{b1}(v) = 1$, $\widehat{p}_{a2}(v) + \widehat{p}_{b2}(v) = 0$.

Define $\widehat{\Theta}_1 = \{v \in \Theta : \widehat{p}_{a1}(v) + \widehat{p}_{b1}(v) = 1\}$. Let us determine how $\widehat{\Theta}_1$ compares with Θ_1 as a result of the changes made in this step. First, note that the buyer's types who previously where assigned contracts that delivered exactly 0, 1, or 2 units, still have their choices available and thus their assignments do not change. Next, we argue that every type $v = (v_a, v_b)$ that previously received total expected quantity in $(1, 2)$ will self-select into some contract that delivers either quantity 1 or 2. Let $\delta = v_a - v_b$. We know that $v_b > v_b'$ for every $v_b' \in I_b(\delta)$. Since $I_b(\delta) \subseteq \widehat{I}_b(\delta)$, we know by Lemma 4 that v has to choose contracts with quantity at least as high as contracts chosen by types in $I_b(\delta)$, i.e. v has to choose contracts that deliver quantity 1 or 2. The argument that the buyer's types who were previously getting expected quantity in $(0, 1)$ will all self-select into contracts that deliver quantity either 0 or 1 is similar.

Thus $\Theta_1 \subseteq \widehat{\Theta}_1$, and all types in $\widehat{\Theta}_1 \backslash \Theta_1$ receive exactly one unit. From (4) in Lemma 5 we know that contribution of types in $\widehat{\Theta}_1 \backslash \Theta_1$ to the costs is zero. Thus we can use Θ_1 in place of $\widehat{\Theta}_1$ when writing down the expression of expected profit from the adjusted mechanism according to formula given in Lemma 5.

The difference in profit between the new and the original mechanisms is

$$
\begin{aligned}
& \sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} \left(U\left(\underline{v}, v_{-i}\right) - \widehat{U}\left(\underline{v}, v_{-i}\right) \right) \left(\underline{v} - c_1\right) f\left(\underline{v}, v_{-i}\right) dv_{-i} \\
& + \int_{\Theta} \left(U\left(v\right) - \widehat{U}\left(v\right) \right) \left(3f\left(v\right) + \sum_{i=a,b} \left(v_i - c_1\right) \frac{\partial f\left(v\right)}{\partial v_i} \right) dv \\
& - \Delta c \int_{\partial\Theta_1} \left(U\left(v\right) - \widehat{U}\left(v\right) \right) \mathbf{n}\left(v\right) f\left(v\right) dv + \Delta c \int_{\Theta_1} \left(U\left(v\right) - \widehat{U}\left(v\right) \right) \left(\sum_{i=a,b} \frac{\partial f\left(v\right)}{\partial v_i} \right) dv \\
= & \sum_{i=a,b} \int_{\underline{v}}^{\overline{v}} \left(U\left(\underline{v}, v_{-i}\right) - \widehat{U}\left(\underline{v}, v_{-i}\right) \right) \left(\underline{v} - c_1 - \Delta c \mathbf{1}_{v \in \Theta_1}\right) f\left(\underline{v}, v_{-i}\right) dv_{-i} \\
& + \int_{\Theta} \left(U\left(v\right) - \widehat{U}\left(v\right) \right) \left[3f\left(v\right) + \sum_{i=a,b} \left(v_i - c_1 - \Delta c \mathbf{1}_{v \in \Theta_1}\right) \frac{\partial f\left(v\right)}{\partial v_i} \right] dv \\
\geq & \ 0
\end{aligned}
$$

where the expression before the equality takes into account $\widehat{U}(v) = U(v)$ for $v \in \overline{\partial\Theta}$; the equality holds because $\widehat{U}(v) = U(v)$ for $v \in \partial\Theta_1 \cap \overline{\partial\Theta}$, and $\widehat{U}(v) = U(v)$ for $v \in \partial\Theta_1 \cap \mathit{interior}(\Theta)$ (since all these types receive exactly one unit); the inequality is due to condition (1) and the fact that $\widehat{U}(v) \leq U(v)$ for all v. ■

Proof of Proposition 2. First note that in a symmetric environment there is no loss for the seller in using symmetric mechanisms.[18]

Next we show that it is without loss for the seller to use mechanisms that ask the buyer to report whether he wants 0, 1, or 2 units, and the difference in valuations $\delta = v_a - v_b$.

By Proposition 1 there is no loss for the seller in offering mechanisms that provide either 0 units ($p_{ij}(v) = 0$ for $i = a, b,\ j = 1, 2$), or 1 unit ($p_{a1}(v) + p_{b1}(v) = 1,\ p_{i2}(v) = 0$ for $i = a, b$), or 2 units ($p_{aj}(v) + p_{bj}(v) = 1$ for $j = 1, 2$) in total. By Lemma 3 buyer with type $(\underline{v}, \underline{v})$ gets 0 units and pays 0. Thus by IC all other buyer's types that receive 0 units pay 0 as well.

[18] See, for example, Section 1 in Maskin and Riley (1984).

Consider types v, $v' \in \Theta$ such that (i) each type gets exactly one unit; (ii) $v_a - v_b = v'_a - v'_b = \delta$ for some δ. Note that

$$\begin{aligned} U(v) &\geq v_a p_{a1}(v') + v_b p_{b1}(v') - T(v') = \delta p_{a1}(v') + v_b - T(v') \\ &= \delta p_{a1}(v') + v'_b - T(v') + (v_b - v'_b) = U(v') + (v_b - v'_b) \end{aligned}$$

where the inequality is due to IC, and the first two equalities make use of (i) and (ii). Similarly

$$U(v') \geq U(v) + (v'_b - v_b).$$

Hence

$$U(v) = U(v') + (v_b - v'_b)$$

i.e. all types with a given difference in valuations δ that receive one unit are indifferent between sending each other's messages. Denote $\overline{\delta} = \overline{v} - \underline{v}$. For every $\delta \in \left[0, \overline{\delta}\right]$ let us pick one specific type v that receives one unit (if there are such types) and assign the contract that he receives to all the other types that were receiving one unit. Let us introduce the following notation: $q_1(\delta) = p_{a1}(v)$, $t_1(\delta) = T(v)$, $u_1(\delta) = \delta q_1(\delta) - t_1(\delta)$. For every $\delta \in \left[-\overline{\delta}, 0\right)$ we define $q_1(\delta)$, $t_1(\delta)$, $u_1(\delta)$ in a symmetric way. Note that these changes do not affect utilities U and set $\Theta_1 = \{v \in \Theta : p_{a1}(v) + p_{b1}(v) = 1\}$.

Similarly note that for every pair of types v, $v' \in \Theta$ such that (i) each type gets exactly two units; (ii) $v_a - v_b = v'_a - v'_b = \delta$ for some $\delta \in \mathbb{R}$, we have

$$U(v) = U(v') + 2(v_b - v'_b)$$

For every $\delta \in \left[0, \overline{\delta}\right]$ let us pick one specific type v that receives two units (if there are such types) and assign the the contract that he receives to all the other types that were receiving two units. Introduce notation: $q_2(\delta) = p_{a1}(v) + p_{a2}(v)$, $t_2(\delta) = T(v)$, $u_2(\delta) = \delta q_2(\delta) - t_2(\delta)$. For every $\delta \in \left[-\overline{\delta}, 0\right)$ we define $q_2(\delta)$, $t_2(\delta)$, $u_2(\delta)$ in a symmetric way. Once again these changes do not affect utilities U and set Θ_1.

By Lemma 5 the expected profit is completely determined by U and set Θ_1, and thus the above changes do not affect the expected profit. Hence, there is no loss for the seller in using mechanisms that ask the buyer to report whether he wants 0, 1, or 2 units, and the difference in valuations.

Next we rewrite Program I using the new notation and taking into account the above observations.

F constraints on p_{ij} imply that $q_1(\delta) \in [0,1]$, $q_2(\delta) \in [0,2]$ for every $\delta \in [0, \overline{\delta}]$. Symmetry of the mechanism implies that buyers with $\delta \geq 0$ must get larger expected quantities of their preferred goods: $q_1(\delta) \geq 1 - q_1(\delta)$ and $q_2(\delta) \geq 2 - q_2(\delta)$. Otherwise it would be profitable for buyer with δ to report $-\delta$. Combine the above constraints with feasibility constraints to get $q_1(\delta) \in [0.5, 1]$, $q_2(\delta) \in [1, 2]$.

IR constraints are automatically satisfied because the null contract at zero price is available.

IC constraints require that each type with a given difference in valuations who decides to purchase a given number of units should find it optimal to report the difference in valuations truthfully. This implies

$$\delta q_j(\delta) - t_j(\delta) \geq \delta q_j(\delta') - t_j(\delta') \text{ for every } \delta, \delta' \in [0, \overline{\delta}], \; j = 1, 2 \tag{5}$$

We can use a standard argument to show that these constraints can be equivalently represented by an envelope formula and monotonicity constraint as stated in Program 2.[19]

[19]It may happen the set of buyer's types who buy j units contains only a subset of differences in the valuations $[0, \overline{v} - \underline{v}]$. In such a case the mechanism can be extended to be defined for all differences in the valuations.

Denote by $g_j(u_1(\delta), u_2(\delta), \delta)$ the measure of types with difference in the valuations δ who buy j units. Then the expected profit is

$$
\begin{aligned}
& E\left[T(v) - \sum_{i=a,b}\sum_{j=1,2} c_j p_{ij}(v)\right] \\
= & \int_{-\overline{\delta}}^{\overline{\delta}} \left(\sum_{j=1,2}\left(t_j(\delta) - \sum_{k=1}^{j} c_k\right) g_j(u_1(\delta), u_2(\delta), \delta)\right) d\delta \\
= & 2\int_{-0}^{\overline{\delta}} \left(\sum_{j=1,2}\left(\delta q_j(\delta) - u_j(\delta) - \sum_{k=1}^{j} c_k\right) g_j(u_1(\delta), u_2(\delta), \delta)\right) d\delta
\end{aligned}
$$

where the first equality uses the fact that the types who buy zero units pay zero, the second equality uses symmetry of the mechanism and definition of $t_j(\delta)$. ■

5.2 Proofs for Section 3.2

Proof of Proposition 3. (i) From Step 1 in the proof of Proposition 1 we know that it is without loss to consider mechanisms where for any given $\delta \geq 0$ the highest possible type $v_b = \overline{v} - \delta$ buys 2 units.

(ii) Note that $q_2(\delta) \in [1, 2]$ is nondecreasing. Thus if $q_2(\delta') = 2$ for some $\delta \in \left[0, \overline{\delta}\right]$, then $q_2(\delta) = 2$ for every $\delta \in \left[\delta', \overline{\delta}\right]$.

If $q_2(\delta) < 2$ for every $\delta \in \left[0, \overline{\delta}\right]$, then adjust the contract for $\overline{\delta}$ as follows: $\widehat{q}_2\left(\overline{\delta}\right) = 2$, $\widehat{t}_2\left(\overline{\delta}\right) = t_2\left(\overline{\delta}\right) + \overline{\delta}\left(2 - q_2\left(\overline{\delta}\right)\right)$. It is easy to see that a such change satisfies IC, does not change the utilities of either types, and does not affect the expected profit since there is only a single type with $\overline{\delta}$: $(\overline{v}, \underline{v}) = \left(\underline{v} + \overline{\delta}, \underline{v}\right)$.

(iii) By Lemma 3 we know that it is without loss to consider mechanisms such that type $(v_a, v_b) = (\underline{v}, \underline{v})$ buys nothing. We will argue that there is a subset of types of Θ, that includes $(\underline{v}, \underline{v})$, which also buy nothing.

Suppose not. Then consider raising all prices $t_1(\delta)$ and $t_2(\delta)$ for every δ by a small $\varepsilon > 0$. Types such that $U(v) \geq \varepsilon$ will continue to make the same purchasing choices as before. Types such that $U(v) < \varepsilon$ will now buy nothing.

Note that $\frac{\partial}{\partial v_i}U(v) = p_i(v) \geq 0$ for a.e. v. Moreover, $\frac{\partial}{\partial v_a}U(v_a, \underline{v})$, $\frac{\partial}{\partial v_b}U(\underline{v}, v_b) \geq 0.5$ for every $v_a, v_b \in [\underline{v}, \overline{v}]$. Thus $\{v \in \Theta : U(v) < \varepsilon\} \subseteq [\underline{v}, \underline{v} + 2\varepsilon]^2$.

Next note that there exists $M > 0$ such that $f(v) < M$ for every $v \in \Theta$. Profit from every type is bounded by $2\overline{v} - c_1 - c_2$. Thus the loss from rasing all prices by a small ε is at most $(2\overline{v} - c_1 - c_2) 4M\varepsilon^2$, while the gain is at least $\varepsilon(1 - 4M\varepsilon^2)$. Hence, such a price adjustment is profitable which gives a contradiction.

(iv) From Step 2 in the proof of Proposition 1 we know that it is without loss to consider mechanisms where for any given δ if we start from the highest possible $v_b = \overline{v} - \delta$ keep reducing v_b, then eventually we pass through a region of v_b that choose one unit, and get to region of v_b that buy nothing. Now restrict attention only to v_b such that $(v_b + \delta, v_b) \in \Theta$, and the result follows. ■

Lemma 6

$$g_1(u_1(\delta), u_2(\delta), \delta) = \begin{cases} \int_{-u_1(\delta)}^{u_1(\delta)-u_2(\delta)} f(v_b + \delta, v_b)\, dv_b & \text{if} \quad \delta \in [0, \delta_1] \\ \int_{\underline{v}}^{u_1(\delta)-u_2(\delta)} f(v_b + \delta, v_b)\, dv_b & \text{if} \quad \delta \in [\delta_1, \delta_2] \\ 0 & \text{if} \quad \delta \in \left[\delta_2, \overline{\delta}\right] \end{cases}$$

$$g_2(u_1(\delta), u_2(\delta), \delta) = \begin{cases} \int_{u_1(\delta)-u_2(\delta)}^{\overline{v}-\delta} f(v_b + \delta, v_b)\, dv_b & \text{if} \quad \delta \in [0, \delta_2] \\ \int_{\underline{v}}^{\overline{v}-\delta} f(v_b + \delta, v_b)\, dv_b & \text{if} \quad \delta \in \left[\delta_2, \overline{\delta}\right] \end{cases}$$

where δ_1, δ_2 are uniquely determined such that $0 \leq \delta_1 \leq \delta_2 \leq \overline{\delta}$, $u_1(\delta_1) + \underline{v} = 0$, and $u_2(\delta_2) - u_1(\delta_2) + \underline{v} = 0$.

Proof. Fix $\delta \geq 0$. All types v_b that buy one unit satisfy $u_1(\delta) \geq u_2(\delta) + v_b$ and $u_1(\delta) + v_b \geq 0$ which can be written as

$$u_1(\delta) - u_2(\delta) \geq v_b \geq -u_1(\delta) \tag{6}$$

From the proof of part (iv) of Proposition 3 we know that such types v_b exist. Since feasible $v_b \geq \underline{v}$, the measure of types that buy one unit for given δ is

$$g_1\left(u_1\left(\delta\right), u_2\left(\delta\right), \delta\right) = \begin{cases} \int_{-u_1(\delta)}^{u_1(\delta)-u_2(\delta)} f\left(v_b+\delta, v_b\right) dv_b & \text{if} \quad -u_1\left(\delta\right) > \underline{v} \\ \int_{\underline{v}}^{u_1(\delta)-u_2(\delta)} f\left(v_b+\delta, v_b\right) dv_b & \text{if} \quad u_1\left(\delta\right) - u_2\left(\delta\right) > \underline{v} \geq -u_1\left(\delta\right) \\ 0 & \text{if} \quad \underline{v} \geq u_1\left(\delta\right) - u_2\left(\delta\right) \end{cases}$$

All types v_b that buy two units satisfy $u_2\left(\delta\right) + v_b \geq u_1\left(\delta\right)$ and $u_2\left(\delta\right) + 2v_b \geq 0$. From part (i) of Proposition 3 we know that the highest possible type $v_b = \overline{v} - \delta$ buys 2 units. Combining all this gives $\overline{v} - \delta \geq v_b \geq \max\left\{u_1\left(\delta\right) - u_2\left(\delta\right), -\frac{1}{2}u_2\left(\delta\right)\right\}$.

Note that (6) implies $u_1\left(\delta\right) - u_2\left(\delta\right) \geq -\frac{1}{2}u_2\left(\delta\right)$. Hence, types v_b that buy two units satisfy

$$\overline{v} - \delta \geq v_b \geq u_1\left(\delta\right) - u_2\left(\delta\right) \tag{7}$$

Since feasible $v_b \geq \underline{v}$, the measure of types that buy two units for given δ is

$$g_2\left(u_1\left(\delta\right), u_2\left(\delta\right), \delta\right) = \begin{cases} \int_{u_1(\delta)-u_2(\delta)}^{\overline{v}-\delta} f\left(v_b+\delta, v_b\right) dv_b & \text{if} \quad u_1\left(\delta\right) - u_2\left(\delta\right) > \underline{v} \\ \int_{\underline{v}}^{\overline{v}-\delta} f\left(v_b+\delta, v_b\right) dv_b & \text{if} \quad \underline{v} \geq u_1\left(\delta\right) - u_2\left(\delta\right) \end{cases}$$

By Lemma 3 the lowest type $\left(\underline{v}, \underline{v}\right)$ buys 0 units at zero price, i.e. $u_1\left(0\right) + \underline{v} \leq 0$. Combining (6) and (7) we have $u_1\left(\delta\right) + \overline{v} - \delta \geq 0$ for every $\delta \in \left[0, \overline{\delta}\right]$ where $\overline{\delta} = \overline{v} - \underline{v}$. Thus $u_1\left(\overline{\delta}\right) + \underline{v} \geq 0$. Also note that $u_1\left(\delta\right) + \underline{v}$ is continuous, and $\frac{d}{d\delta}\left(u_1\left(\delta\right) + \underline{v}\right) = q_1\left(\delta\right) > 0$ for a.e. δ. Thus there there exists unique $\delta_1 \in \left[0, \overline{\delta}\right]$ such that $u_1\left(\delta_1\right) + \underline{v} = 0$.

Using (6) we get $u_2\left(0\right) - u_1\left(0\right) + \underline{v} \leq u_1\left(0\right) + \underline{v}$, and it was previously shown that $u_1\left(0\right) + \underline{v} \leq 0$. Hence $u_2\left(0\right) - u_1\left(0\right) + \underline{v} \leq 0$. Using (7) we get $u_2\left(\overline{\delta}\right) - u_1\left(\overline{\delta}\right) + \underline{v} \geq 0$. Also note that $u_2\left(\delta\right) - u_1\left(\delta\right) + \underline{v}$ is continuous, and $\frac{d}{d\delta}\left(u_2\left(\delta\right) - u_1\left(\delta\right) + \underline{v}\right) = q_2\left(\delta\right) > 0$ for a.e. δ. Thus there there exists unique $\delta_2 \in \left[0, \overline{\delta}\right]$ such that $u_2\left(\delta_2\right) - u_1\left(\delta_2\right) + \underline{v} = 0$. Note that (6) ensures that $\delta_2 \geq \delta_1$.

Hence, $g_1(u_1(\delta), u_2(\delta), \delta)$ and $g_2(u_1(\delta), u_2(\delta), \delta)$ can be written as in the statement of the result. ■

Next we formulate the seller's problem as an optimal control problem and obtain conditions for optimality. The approach is similar to that in Pavlov (2011b) and relies on optimal control tools that can be found for example in Chapter 3.3 in Seierstad and Sydsæter (1987).

$$\max_{(z_j,q_j,u_j)_{j=1,2}} \int_0^{\bar{\delta}} \left(\sum_{j=1,2} \left(\delta q_j(\delta) - u_j(\delta) - \sum_{k=1}^{j} c_k \right) g_j(u_1(\delta), u_2(\delta), \delta) \right) d\delta$$

subject to

Feasibility:	$q_1(\delta) \geq \frac{1}{2}$, $q_2(\delta) \geq 1$	$\underline{\eta}_1(\delta), \underline{\eta}_2(\delta)$
	$1 - q_1(\delta) \geq 0$, $2 - q_2(\delta) \geq 0$	$\overline{\eta}_1(\delta), \overline{\eta}_2(\delta)$
Incentive Compatibility:	$\dot{u}_1(\delta) = q_1(\delta), \dot{u}_2(\delta) = q_2(\delta)$	$\lambda_1(\delta), \lambda_2(\delta)$
	$\dot{q}_1(\delta) = z_1(\delta), \dot{q}_2(\delta) = z_2(\delta)$	$\xi_1(\delta), \xi_2(\delta)$
	$z_1(\delta) \geq 0, z_2(\delta) \geq 0$	$\mu_1(\delta), \mu_2(\delta)$
Transversality conditions:	$q_1(0), q_1(\bar{\delta}), u_1(0), u_1(\bar{\delta}), q_2(0), q_2(\bar{\delta}), u_2(0), u_2(\bar{\delta})$ are free	

Next we derive the necessary conditions for optimality. Form the Lagrangian

$$\begin{aligned} L &= (\delta q_1 - u_1 - c_1) g_1(u_1, u_2, \delta) + (\delta q_2 - u_2 - c_1 - c_2) g_2(u_1, u_2, \delta) \\ &\quad + \lambda_1 q_1 + \xi_1 z_1 + \underline{\eta}_1 \left(q_1 - \frac{1}{2} \right) + \overline{\eta}_1 (1 - q_1) + \mu_1 z_1 \\ &\quad + \lambda_2 q_2 + \xi_2 z_2 + \underline{\eta}_2 (q_2 - 1) + \overline{\eta}_2 (2 - q_2) + \mu_2 z_2 \end{aligned}$$

First we maximize L with respect to z_1 and z_2.

$$\begin{aligned} L^* &= (\delta q_1 - u_1 - c_1) g_1(u_1, u_2, \delta) + (\delta q_2 - u_2 - c_1 - c_2) g_2(u_1, u_2, \delta) \\ &\quad + \lambda_1 q_1 + \underline{\eta}_1 \left(q_1 - \frac{1}{2} \right) + \overline{\eta}_1 (1 - q_1) \\ &\quad + \lambda_2 q_2 + \underline{\eta}_2 (q_2 - 1) + \overline{\eta}_2 (2 - q_2) \end{aligned}$$

with conditions

$$\begin{aligned}
\mu_1 z_1 &= 0,\ \mu_1 = -\xi_1 \geq 0 \text{ and } \dot{q}_1 = z_1 \geq 0 &(8)\\
\mu_2 z_2 &= 0,\ \mu_2 = -\xi_2 \geq 0 \text{ and } \dot{q}_2 = z_2 \geq 0 &(9)
\end{aligned}$$

Next we get a system of Hamiltonian equations:

$$\begin{cases}
\dot{\lambda}_1 = -\frac{\partial L^*}{\partial u_1} = g_1 - (\delta q_1 - u_1 - c_1)\frac{\partial g_1}{\partial u_1} - (\delta q_2 - u_2 - c_1 - c_2)\frac{\partial g_2}{\partial u_1} \\
\dot{\xi}_1 = -\frac{\partial L^*}{\partial q_1} = -\delta g_1 - \lambda_1 - \underline{\eta}_1 + \overline{\eta}_1 \\
\dot{\lambda}_2 = -\frac{\partial L^*}{\partial u_2} = -(\delta q_1 - u_1 - c_1)\frac{\partial g_1}{\partial u_2} + g_2 - (\delta q_2 - u_2 - c_1 - c_2)\frac{\partial g_2}{\partial u_2} \\
\dot{\xi}_2 = -\frac{\partial L^*}{\partial q_2} = -\delta g_2 - \lambda_2 - \underline{\eta}_2 + \overline{\eta}_2
\end{cases} \tag{10}$$

The transversality conditions imply the following boundary requirements for $\lambda_1, \lambda_2, \xi_1, \xi_2$:

$$\begin{aligned}
\lambda_1(0) &= \lambda_1\left(\overline{\delta}\right) = \lambda_2(0) = \lambda_2\left(\overline{\delta}\right) = 0 &(11)\\
\xi_1(0) &= \xi_1\left(\overline{\delta}\right) = \xi_2(0) = \xi_2\left(\overline{\delta}\right) = 0 &(12)
\end{aligned}$$

Co-state variables $\lambda_1, \lambda_2, \xi_1, \xi_2$ are continuous throughout. Moreover, ξ_1 (ξ_2) is equal to zero at the points where the state variable q_1 (q_2) jumps. The remaining conditions are

$$\begin{aligned}
&\underline{\eta}_1\left(q_1 - \frac{1}{2}\right) = 0,\ \underline{\eta}_1 \geq 0 \text{ and } q_1 \geq \frac{1}{2} &(13)\\
&\overline{\eta}_1(1 - q_1) = 0,\ \overline{\eta}_1 \geq 0 \text{ and } q_1 \leq 1 &\\
&\underline{\eta}_2(q_2 - 1) = 0,\ \underline{\eta}_2 \geq 0 \text{ and } q_2 \geq 1 &(14)\\
&\overline{\eta}_2(2 - q_2) = 0,\ \overline{\eta}_2 \geq 0 \text{ and } q_2 \leq 2 &(15)
\end{aligned}$$

Define marginal profit functions as follows:

$$V_j(\delta) = \delta g_j(u_1(\delta), u_2(\delta), \delta) + \lambda_j(\delta) \text{ for } j = 1, 2 \tag{16}$$

Note that $V_j(0) = V_j(\overline{\delta}) = 0$ for $j = 1, 2$. Thus

$$\begin{aligned}
\dot{V}_1 &= 2g_1 + \delta\left(\frac{\partial g_1}{\partial u_1}q_1 + \frac{\partial g_1}{\partial u_2}q_2 + \frac{\partial g_1}{\partial \delta}\right) - (\delta q_1 - u_1 - c_1)\frac{\partial g_1}{\partial u_1} - (\delta q_2 - u_2 - c_1 - c_2)\frac{\partial g_2}{\partial u_1} \\
&= 2g_1 + \delta\frac{\partial g_1}{\partial \delta} + (u_1 + c_1)\frac{\partial g_1}{\partial u_1} + (u_2 + c_2)\frac{\partial g_2}{\partial u_1}
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2 &= 2g_2 + \delta\left(\frac{\partial g_2}{\partial u_1}q_1 + \frac{\partial g_2}{\partial u_2}q_2 + \frac{\partial g_2}{\partial \delta}\right) - (\delta q_1 - u_1 - c_1)\frac{\partial g_1}{\partial u_2} - (\delta q_2 - u_2 - c_1 - c_2)\frac{\partial g_2}{\partial u_2} \\
&= 2g_2 + \delta\frac{\partial g_2}{\partial \delta} + (u_1 + c_1)\frac{\partial g_1}{\partial u_2} + (u_2 + c_2)\frac{\partial g_2}{\partial u_2}
\end{aligned}$$

where the second inequality in both cases uses the fact that $\frac{\partial g_2}{\partial u_1} = \frac{\partial g_1}{\partial u_2}$.

Next we derive the so-called "ironing" necessary conditions for optimality.

Lemma 7 *(i) If q_j is strictly increasing on (δ', δ''), then $V_j(\delta) = 0$ on this interval.*

(ii) If $q_j(\delta) = \frac{j}{2}$ on (δ', δ''), then $\delta' = 0$, $V_j(\delta'') = 0$, $\int_{\delta'}^{\delta''} V_j(\widetilde{\delta})\, d\widetilde{\delta} \leq 0$, and $\int_{\delta}^{\delta''} V_j(\widetilde{\delta})\, d\widetilde{\delta} \leq 0$ for every $\delta \in (\delta', \delta'')$.

(iii) If q_j is a constant in $\left(\frac{j}{2}, j\right)$ on (δ', δ''), then $V_j(\delta') = V_j(\delta'') = 0$, $\int_{\delta'}^{\delta''} V_j(\widetilde{\delta})\, d\widetilde{\delta} = 0$, and $\int_{\delta}^{\delta''} V_j(\widetilde{\delta})\, d\widetilde{\delta} \leq 0$ for every $\delta \in (\delta', \delta'')$.

(iv) If $q_j(\delta) = j$ on (δ', δ''), then $V_j(\delta') = 0$, $\delta'' = \overline{\delta}$, $\int_{\delta'}^{\delta''} V_j(\widetilde{\delta})\, d\widetilde{\delta} \geq 0$, and $\int_{\delta'}^{\delta} V_j(\widetilde{\delta})\, d\widetilde{\delta} \geq 0$ for every $\delta \in (\delta', \delta'')$.

Proof. (i) If q_j is strictly increasing on (δ', δ''), then $\underline{\eta}_j(\delta) = \overline{\eta}_j(\delta) = 0$ and $\mu_j(\delta) = -\xi_j(\delta) = 0$ on (δ', δ''). Thus $\dot{\xi}_1(\delta) = 0$, which implies $V_j(\delta) = 0$ on (δ', δ'').

(ii) Let $j = 1$ (the case of $j = 2$ is similar). Suppose $q_1(\delta) = \frac{1}{2}$ on (δ', δ''). Because of monotonicity of q_1 and constraint $q_1(\delta) \geq \frac{1}{2}$ $\forall\delta$, we must have $\delta' = 0$. The transversality conditions require $\lambda_1(0) = \xi_1(0) = 0$, and we also have $\underline{\eta}_1(\delta) \geq 0$, $\overline{\eta}_1(\delta) = 0$ on $(0, \delta'')$. Thus $\dot{\xi}_1(\delta) = -V_1(\delta) - \underline{\eta}_1(\delta)$, and so $\xi_1(\delta) = -\int_0^{\delta} V_1(\widetilde{\delta})\, d\widetilde{\delta} - \int_0^{\delta} \underline{\eta}_1(\widetilde{\delta})\, d\widetilde{\delta}$ on $(0, \delta'')$.

If $\delta'' = \overline{\delta}$, then $g_1\left(u_1\left(\overline{\delta}\right), u_2\left(\overline{\delta}\right), \overline{\delta}\right) = 0$ and $\lambda_1\left(\overline{\delta}\right) = 0$, and so $V_j\left(\overline{\delta}\right) = 0$. Note that $\xi_1\left(\overline{\delta}\right) = 0$ implies

$$0 = \xi_1\left(\overline{\delta}\right) - \xi_1(0) = -\int_0^{\overline{\delta}} V_1\left(\widetilde{\delta}\right) d\widetilde{\delta} - \int_0^{\overline{\delta}} \underline{\eta}_1\left(\widetilde{\delta}\right) d\widetilde{\delta} \leq -\int_0^{\overline{\delta}} V_1\left(\widetilde{\delta}\right) d\widetilde{\delta}$$

Similarly, $\xi_1(\delta) \leq 0$ for every δ implies

$$0 \leq \xi_1\left(\overline{\delta}\right) - \xi_1(\delta) = -\int_\delta^{\overline{\delta}} V_1\left(\widetilde{\delta}\right) d\widetilde{\delta} - \int_\delta^{\overline{\delta}} \underline{\eta}_1\left(\widetilde{\delta}\right) d\widetilde{\delta} \leq -\int_\delta^{\overline{\delta}} V_1\left(\widetilde{\delta}\right) d\widetilde{\delta}$$

If $\delta'' < \overline{\delta}$, then $\xi_1(\delta'') = 0$ because q_1 changes at δ''. Note that $\xi_1(\delta) \leq 0$ for every δ, $\dot{\xi}_1\left(\delta''_-\right) = -V_1\left(\delta''_-\right) - \underline{\eta}_1\left(\delta''_-\right)$, and $\dot{\xi}_1\left(\delta''_+\right) = -V_1\left(\delta''_+\right) + \overline{\eta}_1\left(\delta''_+\right)$. Since V_1 is continuous at δ'', we have $\dot{\xi}_1\left(\delta''_-\right) \leq -V_1(\delta'') \leq \dot{\xi}_1\left(\delta''_+\right)$. On the other hand $\xi_1(\delta) \leq 0$ and $\xi_1(\delta'') = 0$ imply that $\dot{\xi}_1\left(\delta''_-\right) \geq \dot{\xi}_1\left(\delta''_+\right)$. Thus $\dot{\xi}_1\left(\delta''_-\right) = \dot{\xi}_1\left(\delta''_+\right) = -V_1(\delta'') = 0$. The argument for $\int_{\delta'}^{\delta''} V_j\left(\widetilde{\delta}\right) d\widetilde{\delta} \leq 0$, and $\int_{\delta}^{\delta''} V_j\left(\widetilde{\delta}\right) d\widetilde{\delta} \leq 0$ is similar to the case when $\delta'' = \overline{\delta}$.

The proofs of (iii) and (iv) are similar to the proof of (ii). ■

Calculations for Example 1

Note that

$$g_1\left(u_1(\delta), u_2(\delta), \delta\right) = \begin{cases} 2u_1(\delta) - u_2(\delta) & \text{if} \quad \delta \in [0, \delta_1) \\ u_1(\delta) - u_2(\delta) - c & \text{if} \quad \delta \in (\delta_1, \delta_2] \\ 0 & \text{if} \quad \delta \in (\delta_2, 1] \end{cases}$$

$$g_2\left(u_1(\delta), u_2(\delta), \delta\right) = \begin{cases} c + 1 - \delta + u_2(\delta) - u_1(\delta) & \text{if} \quad \delta \in [0, \delta_2) \\ 1 - \delta & \text{if} \quad \delta \in (\delta_2, 1] \end{cases}$$

where δ_1,δ_2 are such that

$$\begin{aligned} c + u_1(\delta_1) &= 0 \qquad (17) \\ c + u_2(\delta_2) - u_1(\delta_2) &= 0 \end{aligned}$$

Using the earlier calculated general formulas:

$$\dot{V}_1(\delta) = \begin{cases} 6u_1(\delta) - 3u_2(\delta) - c & \text{if} \quad \delta \in [0, \delta_1) \\ 3u_1(\delta) - 3u_2(\delta) - 3c & \text{if} \quad \delta \in (\delta_1, \delta_2] \\ 0 & \text{if} \quad \delta \in (\delta_2, 1] \end{cases} \tag{18}$$

$$\ddot{V}_1(\delta) = \begin{cases} 6q_1(\delta) - 3q_2(\delta) & \text{if} \quad \delta \in [0, \delta_1) \\ 3q_1(\delta) - 3q_2(\delta) & \text{if} \quad \delta \in (\delta_1, \delta_2] \\ 0 & \text{if} \quad \delta \in (\delta_2, 1] \end{cases}$$

Hence V_1 (weakly) concave on $(\delta_1, \delta_2]$ and is ambiguous on $[0, \delta_1)$. Notice that $\dot{V}_1$ is discontinuous at δ_1 since $c > 0$:

$$\begin{aligned} \dot{V}_1(\delta_{1-}) &= 6u_1(\delta_1) - 3u_2(\delta_1) - c = 3u_1(\delta_1) - 3u_2(\delta_1) - 4c \\ &\leq 3u_1(\delta_1) - 3u_2(\delta_1) - 3c = \dot{V}_1(\delta_{1+}) \end{aligned}$$

Also note that $\dot{V}_1$ is continuous at δ_2:

$$\dot{V}_1(\delta_{2-}) = 3u_1(\delta_2) - 3u_2(\delta_2) - 3c = 0 = \dot{V}_1(\delta_{2+})$$

Next note that

$$\dot{V}_2(\delta) = \begin{cases} 3c + 2 + 3(u_2(\delta) - u_1(\delta)) - 3\delta & \text{if} \quad \delta \in [0, \delta_2) \\ 2 - 3\delta & \text{if} \quad \delta \in (\delta_2, 1] \end{cases}$$

$$\ddot{V}_2(\delta) = \begin{cases} 3(q_2(\delta) - q_1(\delta)) - 3 & \text{if} \quad \delta \in [0, \delta_2) \\ -3 & \text{if} \quad \delta \in (\delta_2, 1] \end{cases}$$

Hence V_2 is concave on $(\delta_2, 1]$ and is ambiguous on $[0, \delta_2)$. Notice that $\dot{V}_2$ is continuous at δ_2:

$$\dot{V}_2(\delta_{2-}) = 3c + 2 + 3(u_2(\delta_2) - u_1(\delta_2)) - 3\delta_2 = 2 - 3\delta_2 = \dot{V}_2(\delta_{2+})$$

Recall that $V_j(0) = V_j(1) = 0$ for $j = 1, 2$. Note that

$$\begin{aligned} 0 &= V_2(1) - V_2(0) = \int_0^{\delta_2} \dot{V}_2(\delta)\, d\delta + \int_{\delta_2}^1 \dot{V}_2(\delta)\, d\delta = -\int_0^{\delta_2} \ddot{V}_2(\delta)\, \delta d\delta - \int_{\delta_2}^1 \ddot{V}_2(\delta)\, \delta d\delta + \dot{V}_2(1) \\ &= -\int_0^{\delta_2} 3(q_2(\delta) - q_1(\delta))\, \delta d\delta + \int_0^1 3\delta d\delta - 1 = -\int_0^{\delta_2} 3(q_2(\delta) - q_1(\delta))\, \delta d\delta + \frac{1}{2} \end{aligned}$$

or

$$\int_0^{\delta_2} 3(q_2(\delta) - q_1(\delta))\, \delta d\delta = \frac{1}{2} \tag{19}$$

Next note that

$$\begin{aligned} 0 &= V_1(1) - V_1(0) = \int_0^{\delta_1} \dot{V}_1(\delta)\, d\delta + \int_{\delta_1}^{\delta_2} \dot{V}_1(\delta)\, d\delta \\ &= \left(\dot{V}_1(\delta_{1-}) - \dot{V}_1(\delta_{1+})\right)\delta_1 - \int_0^{\delta_1} \ddot{V}_1(\delta)\, \delta d\delta - \int_{\delta_1}^{\delta_2} \ddot{V}_1(\delta)\, \delta d\delta \\ &= -c\delta_1 - \int_0^{\delta_1} 3q_1(\delta)\, \delta d\delta + 3\int_0^{\delta_2} (q_2(\delta) - q_1(\delta))\, \delta d\delta \end{aligned}$$

Using (6) we get

$$c\delta_1 + \int_0^{\delta_1} 3q_1(\delta)\, \delta d\delta = \frac{1}{2} \tag{20}$$

By Proposition 3 we know that $q_2(\delta) = 2$ for δ sufficiently high. Let in the optimal mechanism $q_2(\delta) = 2$ whenever $\delta \in (t, 1]$.

Lemma 8 *(i) There exists $t \in [0, \delta_2)$ such that $q_2(\delta) = 2$ iff $\delta \in (t, 1]$.*

(ii) If $t > 0$, then $V_2(\delta) = 0$ for every $\delta \in [0, t)$.

Proof. (i) By (19) we have $\frac{1}{2} \le \int_0^{\delta_2} 3\left(2 - \frac{1}{2}\right)\delta d\delta = \frac{9}{4}\delta_2^2$. Thus $\delta_2 \ge \frac{\sqrt{2}}{3}$.

Note that for $\delta \in (\delta_2, 1]$ we have

$$V_2(\delta) = V_2(1) - \int_\delta^1 \dot{V}_2\left(\widetilde{\delta}\right) d\widetilde{\delta} = -\int_\delta^1 \left(2 - 3\widetilde{\delta}\right) d\widetilde{\delta} = \frac{1}{2}(3\delta - 1)(1 - \delta)$$

Since $\delta_2 \geq \frac{\sqrt{2}}{3}$, we have $V_2(\delta) > 0$ for every $\delta \in (\delta_2, 1)$. Thus by Lemma 7 $q_2(\delta) = 2$ on $(\delta_2, 1)$ and there exists $t \in [0, \delta_2)$ such that $q_2(\delta) = 2$ iff $\delta \in (t, 1]$.

(ii) Suppose $q_2(\delta) = k < 2$ iff $\delta \in (t', t'')$. By Lemma 7 we have $V_2(t') = V_2(t'') = 0$.

Suppose first that $V_2(\delta) \leq 0$ for every $\delta \in (t', t'')$ with inequality being strict at some point. Then by Lemma 7 we must have $k = 1$. Since by (i) we have $t'' < \delta_2$, we have $\ddot{V}_2(\delta) = 3(q_2(\delta) - q_1(\delta)) - 3 = -3q_1(\delta) < 0$ for every $\delta \in (t', t'')$. Thus V_2 is concave on (t', t''), which together with $V_2(t') = V_2(t'') = 0$ implies $V_2(\delta) = 0$ for every $\delta \in (t', t'')$. But this contradicts the assumption that V_2 is strictly negative at some point.

Suppose that V_2 is strictly positive at some point in $\delta^* \in (t', t'')$. Note that by Lemma 7 it is impossible that $V_2(\delta) \geq 0$ for every $\delta \in (t', t'')$, and there must exist $t^* \in (\delta^*, t'')$ such that $V_2(t^*) = 0$ and $V_2(\delta) < 0$ for some $\delta \in (t^*, t'')$. Note that $\ddot{V}_2(\delta) = 3(k - q_1(\delta)) - 3$ is nonincreasing on (t', t'').

Let $\ddot{V}_2(\delta) \leq 0$ for every $\delta \in (t^*, t'')$. Then V_2 is concave on (t^*, t''), which together with $V_2(t^*) = V_2(t'') = 0$ implies $V_2(\delta) = 0$ for every $\delta \in (t^*, t'')$. But this contradicts the assumption that V_2 is strictly negative at some point on (t^*, t'').

Now suppose that $\ddot{V}_2$ changes sign on (t^*, t''), and thus $\ddot{V}_2(\delta) > 0$ for every $\delta \in (t', t^*)$. Then V_2 is strictly convex on (t', t^*), which together with $V_2(t') = V_2(t^*) = 0$ implies $V_2(\delta) < 0$ for every $\delta \in (t', t^*)$. But this contradicts the assumption that V_2 is strictly positive at some point on (t', t^*). ■

Case 1 $q_2(\delta) = 2$ for every $\delta \in [0, 1]$

Since $V_2(0) = 0$ and V_2 is convex on $(0, \delta_2)$, by Lemma 7 the condition for optimality of $q_2(\delta) = 2$ for every $\delta \in [0, 1]$ simplifies to $\dot{V}_2(0) \geq 0$, or

$$3c + 2 + 3(u_2(0) - u_1(0)) \geq 0 \tag{21}$$

From (18) it follows that V_1 is concave on $(0, \delta_1)$, and concave on (δ_1, δ_2). Also recall that $\dot{V}_1(\delta_{1-}) \leq \dot{V}_2(\delta_{1+})$. Notice also that (17) implies that $\dot{V}_1(\delta_2) = 0$, and so V_1 is increasing on

(δ_1, δ_2). Since $V_1(0) = V_1(\delta_2) = 0$, we have that V_1 crosses zero on $(0, \delta_1)$ at most once, and this crossing is from above. It follows from Lemma 7 that q_1 is constant on $(0, \delta_2)$.

Denote $q_1(\delta) = \alpha$ for every $\delta \in [0, 1]$. Conditions (17), (19) and (20) become

$$
\begin{aligned}
& c + u_1(0) + \alpha\delta_1 = 0 \\
& c + u_2(0) - u_1(0) + (2-\alpha)\,\delta_2 = 0 \\
& \frac{3}{2}(2-\alpha)\,\delta_2^2 = \frac{1}{2} \\
& c\delta_1 + \frac{3\alpha}{2}\delta_1^2 = \frac{1}{2}
\end{aligned}
$$

The solution is

$$
\begin{aligned}
& \delta_1 = \frac{1}{3\alpha}\left(\sqrt{c^2 + 3\alpha} - c\right),\ \delta_2 = \frac{1}{\sqrt{3(2-\alpha)}} \\
& u_1(0) = -\frac{1}{3}\sqrt{c^2 + 3\alpha} - \frac{2}{3}c,\ u_2(0) = u_1(0) - c - \frac{1}{3}\sqrt{3(2-\alpha)}
\end{aligned}
$$

Note that condition (21) simplifies to $\alpha \geq \frac{2}{3}$.

Next note that

$$
\begin{aligned}
& \int_0^{\delta_2} V_1(\delta)\, d\delta = -\int_0^{\delta_1} \dot{V}_1(\delta)\,\delta d\delta - \int_{\delta_1}^{\delta_2} \dot{V}_1(\delta)\,\delta d\delta \\
= \;& -\frac{1}{2}\left(\dot{V}_1(\delta_{1-}) - \dot{V}_1(\delta_{1+})\right)\delta_1^2 + \frac{1}{2}\int_0^{\delta_1} \ddot{V}_1(\delta)\,\delta^2 d\delta + \frac{1}{2}\int_{\delta_1}^{\delta_2} \ddot{V}_1(\delta)\,\delta^2 d\delta \\
= \;& \frac{1}{2}c\delta_1^2 + \frac{1}{6}(6\alpha - 6)\,\delta_1^3 + \frac{1}{6}(3\alpha - 6)\left(\delta_2^3 - \delta_1^3\right) \\
= \;& \frac{1}{6}\left(\frac{\sqrt{c^2 + 3\alpha} + 2c}{\left(\sqrt{c^2 + 3\alpha} + c\right)^2} - \frac{1}{\sqrt{3(2-\alpha)}}\right)
\end{aligned}
$$

where the first equality is due to $V_1(\delta_2) = 0$ and continuity of V_1; the second equality is due to $\dot{V}_1(\delta_2) = 0$ and continuity of $\dot{V}_1$ everywhere except at δ_1; the third equality uses formulas for $\ddot{V}_1$ and the facts that $q_1(\delta) = \alpha$ and $q_2(\delta) = 2$ on $(0, \delta_2)$; the fourth equality comes from substituting expressions for δ_1 and δ_2.

Denote the resulting expression by $\Phi(c,\alpha)$. Note that

$$\frac{d}{dc}\left(\frac{\sqrt{c^2+3\alpha}+2c}{\left(\sqrt{c^2+3\alpha}+c\right)^2}\right)=\frac{-3c}{\sqrt{c^2+3\alpha}\left(\sqrt{c^2+3\alpha}+c\right)^2} \tag{22}$$

and

$$\frac{\sqrt{c^2+3\alpha}+2c}{\left(\sqrt{c^2+3\alpha}+c\right)^2}=\frac{1}{\sqrt{c^2+3\alpha}+c}+\frac{c}{\left(\sqrt{c^2+3\alpha}+c\right)^2} \tag{23}$$

Thus Φ is strictly decreasing in α and in c for $c>0$, $\alpha\in\left[\frac{1}{2},1\right]$. Thus there exists a strictly decreasing function $\alpha(c)$ implicitly defined by $\Phi(c,\alpha)=0$, with $\alpha(0)=1$ (since $\Phi(0,1)=0$) and $\alpha(1)=\frac{2}{3}$ (since $\Phi\left(1,\frac{2}{3}\right)=0$).

Thus the optimality conditions of Lemma 7 are satisfied when $\alpha=\alpha(c)$ for every $c\in(0,1]$. The price for $q_1=\alpha(c)$ and $q_2=2$ are $T_1=-u_1(0)$ and $T_2=-u_2(0)$, respectively, evaluated at $\alpha=\alpha(c)$.

The expected profit is

$$\begin{aligned}
&\Pr\{\alpha|v_a-v_b|+\min\{v_a,v_b\}-T_1\geq 0\}T_1\\
&+\Pr\{2\max\{v_a,v_b\}-T_2\geq\alpha|v_a-v_b|+\min\{v_a,v_b\}-T_1\}(T_2-T_1-c)\\
=&\left(1-\frac{1}{9\alpha}\left(\sqrt{c^2+3\alpha}-c\right)^2\right)\left(\frac{1}{3}\sqrt{c^2+3\alpha}+\frac{2}{3}c\right)+\frac{2}{3}\left(\frac{1}{3}\sqrt{3(2-\alpha)}\right)
\end{aligned}$$

Case 2 $q_2(\delta)=2$ only $\forall\delta\in(t,1]$ where $t\in(0,\delta_2)$.

By Lemma 8 on $[0,t)$ we have

$$\dot{V}_2(\delta)=3c+2+3(u_2(\delta)-u_1(\delta))-3\delta=0$$

and thus

$$\ddot{V}_2(\delta)=3(q_2(\delta)-q_1(\delta)-1)=0 \tag{24}$$

$\dot{V}_2(0)=0$ implies

$$u_2(0) - u_1(0) = -c - \frac{2}{3} \tag{25}$$

Since $V_2(t) = 0$ and V_2 is convex on (t, δ_2), by Lemma 7 the condition for optimality of $q_2(\delta) = 2$ for every $\delta \in [0,1]$ simplifies to $\dot{V}_2(t) \geq 0$. This condition is satisfied since $\dot{V}_2(\delta) = 0$ on $[0, t)$ and $\dot{V}_2$ is continuous.

Note that on $[0, \min\{t, \delta_1\})$ we have

$$\ddot{V}_1(\delta) = 6q_1(\delta) - 3q_2(\delta) = 3q_2(\delta) - 6 < 0$$

where the second equality follows from (24). Hence V_1 is concave on $[0, \min\{t, \delta_1\})$, and by Lemma 7 it must be that q_1 is constant. Denote $q_1(\delta) = \alpha$ on $[0, \min\{t, \delta_1\})$, and note that $q_2(\delta) = \alpha + 1$ on this interval.

If $\delta_1 \leq t$, then using an argument identical to Case 1 we find that q_1 is constant on $(0, \delta_2)$. If $t < \delta_1$, then the facts that $\dot{V}_1$ is continuous at t and V_1 is weakly concave on (t, δ_1) again imply that q_1 is constant on $(0, \delta_2)$.

Thus like in Case 1 we have

$$\delta_1 = \frac{1}{3\alpha}\left(\sqrt{c^2 + 3\alpha} - c\right), \; u_1(0) = -\frac{1}{3}\sqrt{c^2 + 3\alpha} - \frac{2}{3}c$$

The second part of condition (17) and condition (19) in this case are

$$\begin{aligned} & c + u_2(0) + (1+\alpha)t + 2(\delta_2 - t) - u_1(0) - \alpha\delta_2 = 0 \\ & \frac{3}{2}t^2 + \frac{3}{2}(2-\alpha)\left(\delta_2^2 - t^2\right) = \frac{1}{2} \end{aligned}$$

Using (25) we find that

$$t = \frac{2}{3} - \frac{1}{3}\sqrt{\frac{2-\alpha}{1-\alpha}}, \; \delta_2 = \frac{2}{3} - \frac{1}{3}\sqrt{\frac{1-\alpha}{2-\alpha}}$$

Note that $t \geq 0$ iff $\alpha \leq \frac{2}{3}$.

Next note that

$$\int_0^{\delta_2} V_1(\delta)\,d\delta = -\frac{1}{2}\left(\dot{V}_1(\delta_{1-}) - \dot{V}_1(\delta_{1+})\right)\delta_1^2 + \frac{1}{2}\int_0^{\delta_1}\ddot{V}_1(\delta)\,\delta^2 d\delta + \frac{1}{2}\int_{\delta_1}^{\delta_2}\ddot{V}_1(\delta)\,\delta^2 d\delta$$

$$= \frac{1}{2}c\delta_1^2 + \frac{1}{2}\int_0^{\delta_1} 6\alpha\delta^2 d\delta + \frac{1}{2}\int_{\delta_1}^{\delta_2} 3\alpha\delta^2 d\delta - \frac{1}{2}\int_0^t 3(1+\alpha)\,\delta^2 d\delta - \frac{1}{2}\int_t^{\delta_2} 3(2)\,\delta^2 d\delta$$

$$= \frac{1}{2}c\delta_1^2 + \alpha\delta_1^3 + \frac{\alpha}{2}\left(\delta_2^3 - \delta_1^3\right) - \frac{1+\alpha}{2}t^3 - \left(\delta_2^3 - t^3\right)$$

$$= \frac{\sqrt{c^2+3\alpha}+2c}{6\left(\sqrt{c^2+3\alpha}+c\right)^2} - \frac{1}{54}\left(2 + \sqrt{\frac{2-\alpha}{1-\alpha}} + \sqrt{\frac{1-\alpha}{2-\alpha}}\right)$$

where the first equality is obtained in the same way as in Case 1; the second equality uses formulas for $\ddot{V}_1$ and the facts that $q_1(\delta) = \alpha$ on $(0, \delta_2)$ and $q_2(\delta) = 1 + \alpha$ on $(0, t)$ and $= 2$ on $(t, 1)$; the fourth equality comes from substituting expressions for δ_1 and δ_2.

Denote the resulting expression by $\Psi(c, \alpha)$. Note that $\frac{2-\alpha}{1-\alpha}$ is increasing in α, $\sqrt{\frac{2-\alpha}{1-\alpha}} > 1$ for $\alpha \geq \frac{1}{2}$, and $x + \frac{1}{x}$ is increasing for $x > 1$. Thus, using (22) and (23), Ψ is strictly decreasing in α and in c for $c > 0$, $\alpha \in \left[\frac{1}{2}, 1\right]$. Thus there exists a strictly decreasing function $\alpha(c)$ implicitly defined by $\Psi(c, \alpha) = 0$, with $\alpha(1) = \frac{2}{3}$ (since $\Psi\left(1, \frac{2}{3}\right) = 0$) and $\alpha(c^*) = \frac{1}{2}$ for $c^* \approx 1.22$ such that $\Psi\left(c^*, \frac{1}{2}\right) = 0$). Note that $\Psi\left(c, \frac{1}{2}\right) < 0$ for every $c > c^*$, and thus we denote $\alpha(c) = \frac{1}{2}$ for this range of c.

Thus the optimality conditions of Lemma 7 are satisfied when $\alpha = \alpha(c)$ for every $c \geq 1$. The price for $q_1 = \alpha(c)$ is $T_1 = -u_1(0)$, for $q_2 = 1 + \alpha(c)$ is $\underline{T}_2 = -u_2(0)$, and for $q_2 = 2$ is $\overline{T}_2 = -u_2(0) + tq_2(t_+) - \int_0^t q_2(\delta)\,d\delta = -u_2(0) + (1-\alpha)\,t$, evaluated at $\alpha = \alpha(c)$.

The expected profit is

$$
\begin{aligned}
&\Pr\left\{\alpha\left|v_a-v_b\right|+\min\left\{v_a,v_b\right\}-T_1\geq 0\right\}T_1\\
&+\Pr\left\{\begin{array}{c}(1+\alpha)\left|v_a-v_b\right|+2\min\left\{v_a,v_b\right\}-\underline{T}_2\geq\\ \max\left\{2\max\left\{v_a,v_b\right\}-\overline{T}_2,\alpha\left|v_a-v_b\right|+\min\left\{v_a,v_b\right\}-T_1\right\}\end{array}\right\}\left(\underline{T}_2-T_1-c\right)\\
&+\Pr\left\{\begin{array}{c}2\max\left\{v_a,v_b\right\}-\overline{T}_2\geq\\ \max\left\{(1+\alpha)\left|v_a-v_b\right|+2\min\left\{v_a,v_b\right\}-\underline{T}_2,\alpha\left|v_a-v_b\right|+\min\left\{v_a,v_b\right\}\right\}\end{array}\right\}\left(\overline{T}_2-T_1-c\right)\\
=\;&\left(1-\frac{1}{9\alpha}\left(\sqrt{c^2+3\alpha}-c\right)^2\right)\left(\frac{1}{3}\sqrt{c^2+3\alpha}+\frac{2}{3}c\right)+\frac{2}{9}\left(2-\sqrt{\frac{2-\alpha}{1-\alpha}}\right)\left(\frac{2}{3}\right)\\
&+\frac{2}{9}\left(1+\sqrt{\frac{2-\alpha}{1-\alpha}}\right)\left(\frac{2}{3}+(1-\alpha)\left(\frac{2}{3}-\frac{1}{3}\sqrt{\frac{2-\alpha}{1-\alpha}}\right)\right)
\end{aligned}
$$

5.3 Proofs for Section 3.3

Proof of Proposition 4. Suppose the optimal joint selling solution offers: $(0,0)$ at a price 0; $(1,0)$ and $(0,1)$ at a price x; $(2,0)$ and $(0,2)$ at a price y. Clearly $x>0$, and also $y-x>x$ because otherwise no one would choose to buy 1 unit.

Consider the following mechanisms for separate selling of units 1 and 2. To sell the first unit the seller offers: $(0,0)$ at a price 0; $(1,0)$ and $(0,1)$ at a price x. To sell the second unit the seller offers: $(0,0)$ at a price 0; $(1,0)$ and $(0,1)$ at a price $y-x$.

Note that the seller obtains the same expected profit from separate selling as from the optimal joint selling. Since the profit from separate selling of the two units cannot exceed the optimal profit from joint selling it must be that the above mechanisms for separate selling are optimal. ■

Calculations for Example 3

As before we can focus on symmetric mechanism and just consider the case $v_a\geq v_b$. For every v we can find type $\widetilde{v}$ such that $\widetilde{v}_a=1$, and such that there exists $s\in[0,1]$ such that $v=s\widetilde{v}$. Let us change variables from (v_a,v_b) to $(s,\widetilde{v}_b)$. The Jacobian transform is s.

The expected profit is

$$\begin{aligned}
& \int_0^1 \int_0^{v_a} \left(\sum_{i=a,b} \sum_{j=1,2} (v_i - c_j)\, p_{ij}(v) - U(v) \right) dv_b dv_a \\
= & \int_0^1 \int_0^1 \left(\sum_{j=1,2} \left((s - c_j)\, p_{aj}(s, s\widetilde{v}_b) + (s\widetilde{v}_b - c_j)\, p_{bj}(s, s\widetilde{v}_b) \right) - U(s, s\widetilde{v}_b) \right) s ds d\widetilde{v}_b
\end{aligned}$$

The envelope condition implies

$$\begin{aligned}
U(s, s\widetilde{v}_b) &= U(0) + \int_0^s \frac{d}{ds} \left[U(\widetilde{s}, \widetilde{s}\widetilde{v}_b) \right] d\widetilde{s} = U(0) + \int_0^s \left(\frac{\partial U(\widetilde{s}, \widetilde{s}\widetilde{v}_b)}{\partial v_a} + \widetilde{v}_b \frac{\partial U(\widetilde{s}, \widetilde{s}\widetilde{v}_b)}{\partial v_b} \right) d\widetilde{s} \\
&= U(0) + \int_0^s \left(\sum_{j=1,2} \left(p_{aj}(\widetilde{s}, \widetilde{s}\widetilde{v}_b) + \widetilde{v}_b p_{bj}(\widetilde{s}, \widetilde{s}\widetilde{v}_b) \right) \right) d\widetilde{s}
\end{aligned}$$

Thus

$$\begin{aligned}
& \int_0^1 U(s, s\widetilde{v}_b)\, s ds \\
= & U(0) \left(\frac{1}{2} \int_0^1 ds^2 \right) + \left[\left(\int_0^s \left(\sum_{j=1,2} \left(p_{aj}(\widetilde{s}, \widetilde{s}\widetilde{v}_b) + \widetilde{v}_b p_{bj}(\widetilde{s}, \widetilde{s}\widetilde{v}_b) \right) \right) d\widetilde{s} \right) \left(\frac{1}{2} \int_0^s d\widetilde{s}^2 \right) \right]_0^1 \\
& - \int_0^1 \left(\sum_{j=1,2} \left(p_{aj}(s, s\widetilde{v}_b) + \widetilde{v}_b p_{bj}(s, s\widetilde{v}_b) \right) \right) \left(\frac{1}{2} \int_0^s d\widetilde{s}^2 \right) ds \\
= & \frac{1}{2} U(0) + \int_0^1 \left(\sum_{j=1,2} \left(p_{aj}(s, s\widetilde{v}_b) + \widetilde{v}_b p_{bj}(s, s\widetilde{v}_b) \right) \right) \frac{1 - s^2}{2} ds
\end{aligned}$$

Hence the expected profit is

$$\begin{aligned}
& \int_0^1 \int_0^1 \left(\sum_{j=1,2} \left(\left(s - \frac{1 - s^2}{2s} - c_j \right) p_{aj}(s, s\widetilde{v}_b) + \left(\widetilde{v}_b \left(s - \frac{1 - s^2}{2s} \right) - c_j \right) p_{bj}(s, s\widetilde{v}_b) \right) \right) s ds d\widetilde{v}_b \\
& - \frac{1}{2} U(0)
\end{aligned}$$

Pointwise maximization yields for $j = 1, 2$: $p_{bj}(s, s\widetilde{v}_b) = 0$; and $p_{aj}(s, s\widetilde{v}_b) = 0$ if $s \left(1 - \frac{1 - s^2}{2s^2} \right) - c_j < 0$, $p_{aj}(s, s\widetilde{v}_b) = 1$ if $s \left(1 - \frac{1 - s^2}{2s^2} \right) - c_j > 0$. Rearranging we get

$p_{aj}(s, s\widetilde{v}_b) = 0$ if $s < \frac{1}{3}\left(\sqrt{c_j^2+3}+c_j\right)$, $p_{aj}(s, s\widetilde{v}_b) = 1$ if $s \geq \frac{1}{3}\left(\sqrt{c_j^2+3}+c_j\right)$.

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