Straight Talk

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Abstract

We revisit the Sender-Receiver game of Crawford and Sobel (1982), and examine whether allowing for long cheap talk increases the set of payoffs. We show that it does, for biases in the range $[1/8, 1/\sqrt{8}]$, and explicitly derive the best equilibrium within some class. We show that the payoff increases with the length of the cheap talk phase, although there is no discontinuity at infinity. Because only finitely many messages (and two rounds) suffice for lower biases, this shows that the number of messages necessary to implement the best equilibrium is not increasing in the congruence of the players' preferences, unlike what the static cheap talk game suggests.

Keywords: Long cheap talk, signaling.

JEL codes: C72, C73

1 Introduction

Most economists regard long cheap talk the way they consider tightrope walking: admirable, but knife-edge. These examples, showing how long communication improves efficiency, impress more by their ingenuity than by their relevance. Meanwhile, intuitive comparative statics already obtain from one-round communication. For instance, the more aligned the players' preferences, the more nontrivial equilibrium messages get exchanged.

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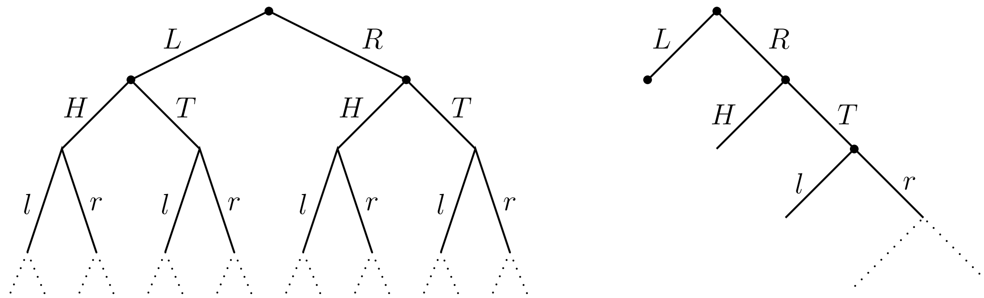

Figure 1: Canonical structure of long talk (left) vs. straight talk (right)

Yet, this paper establishes that long cheap talk isn't an artifact of abstract games reversed-engineered by facetious theorists. Instead, it arises in what is arguably the canonical model of cheap talk, as defined by Crawford and Sobel (1982, hereafter CS). Krishna and Morgan (2004, hereafter KM) already show that adding a round of communication improves upon the partitional equilibria of CS. We show that two rounds isn't enough. More precisely, for intermediate biases, increasing the length of communication results in higher expected payoffs, with no a priori bound on the duration of this communication. Because few messages are needed for low biases, it follows that the relationship between the duration and the complexity of communication on one hand, and the alignment of preferences on the other, isn't as simple as portrayed by the literature.[1]

These seemingly sweeping conclusions come with two caveats. First, they are confined to the uniform-quadratic framework that is a leading, but very special, case of CS. Second, they apply to a class of equilibria that might not be without loss. *Straight talk equilibria* involve an alternation of jointly controlled lotteries (j.c.l.) with two outcomes only, with binary messages sent by the sender. *Per se*, this is without loss of generality, given Aumann and Hart's result.[2] The restriction lies in the assumption that one of the two messages, and one of the outcomes of the j.c.l., lead to termination of the communication phase. See Figure 1, which elucidates the name straight talk: the game tree is as straight as possible.

Not all equilibria are straight talk equilibria, as we show by example in Section 2. Yet, all equilibria considered in the literature on cheap talk can be described as straight talk equilibria. This is true, in particular, for one of the two equilibria constructed by KM –the

[1] Arguably, there is something arbitrary about using the number of messages as a measure for the complexity of communication. The number of possible actions by the receiver is a better measure. In the equilibria we consider, this distinction is irrelevant, and the number of possible actions grows with the number of rounds.

[2] Yet, their results do not apply to our framework, formally speaking, since the game of Crawford and Sobel (1982) involves infinite type and action spaces.

one referred to as monotonic–, which is shown to be optimal for low biases (below one-eighth) by Goltsman, Hörner, Pavlov and Squintani (2009, hereafter GHPS). Furthermore, as we show for the case of two-round communication, no other equilibrium performs better, independent of the size of the bias.[3] Given that GHPS settles the problem of long cheap talk for low biases, we focus on biases above one-eighth.

However, non-straight talk equilibria have been studied in other contexts. The famous example of Forges (1990) does not involve straight talk, as some j.c.l. are performed, which are followed by further communication, independent of the binary outcome. On the other hand, all the other examples mentioned by Aumann and Hart involve straight talk.

The restriction to straight talk equilibria allows for an explicit characterization of the set of equilibria. To simplify, we describe them here for $b > 1/4$ (an additional twist can occur for $b \in (1/8, 1/4)$). In the first stage, a middle interval of types (say, $[x, z]$, with $0 < x < z < 1$) send a common message that separates themselves from the remaining types. If this message is sent, communication stops. If not, and the outcome of the j.c.l. leads to further communication, a second interval of types send a common message. This interval of types is either of the kind $[0, y]$, for some $y \leq x$, or $[z, y]$, for some $y \in [z, 1]$. Hence, it is a lower interval (we say, a lower cut) of one of the remaining intervals. Again, this message ends communication. If not, and again the outcome of the j.c.l. leads to further communication, a further lower cut (at either the bottom or the upper interval of types remaining) obtains. Etc. The non-monotonic equilibrium defined by KM is a special case, in which communication ends after the second round of message, which reveals whether the state is in $[0, x]$ or $[z, 1]$.

Of course, equilibrium imposes restrictions on the values of these cuts. Yet, we show that nontrivial straight talk equilibria exist for any length of the horizon (Theorem 1), provided $b \leq \frac{1}{\sqrt{8}}$. On the other hand, no straight talk equilibrium (beside babbling) exists for larger biases.

Further, not all these cuts are equally desirable. Cuts in the lower interval (*e.g.*, in $[0, x]$) are Pareto-dominated by straight-talk equilibria of the same duration that do not involve such cuts. Furthermore, the best straight talk equilibrium with $T + 1$ rounds of cheap talk improves on the best equilibrium with T rounds of cheap talk (Theorem 2).

One of the benefits of the focus on straight talk is that it allows a simple description of the limiting game, as the number of rounds grows without bound, in terms of an auxiliary continuous-time stopping game, introduced in Section 4. The duration of this game can be

[3]We do not know whether the result holds for an arbitrary number of rounds, although some of the results in Section 2 provide steps in this direction.

normalized to one, with the Receiver taking an action at this end time. The j.c.l. is replaced by a stopping time τ, whose distribution is part of the equilibrium specification. The Sender chooses another stopping time, as a function of his type. Stopping the (communication) game at a given time is equivalent to sending the message that the type is in a particular subset of the unit interval. (The map from time to subset is part of the equilibrium specification.) If τ realizes before the Sender's stopping time, all the Receiver learns is that the Sender's type is one of those for which the stopping time is indeed larger than τ. We show that the best equilibrium in this game captures the limiting behavior of the best equilibrium with finite-length straight talk, as the number of rounds goes to infinity. Conversely, equilibria of the continuous-time game are limits of equilibria with finite-length straight talk. The best continuous-time equilibrium can be described analytically to a large extent.

Related Literature: The most relevant papers, namely CS, KM and GHPS, have already been mentioned. KM shows how two rounds of messages allow to improve efficiency upon partitional equilibria, and stretch the range of biases over which non-babbling equilibria exist from $[0, 1/4]$ to $[0, 1/\sqrt{8}]$. GHPS show that KM's first example (monotonic equilibrium), which applies for biases no larger than 1/4, is actually optimal for $b \leq 1/8$. For larger biases, however, all that GHPS can show is that the mediation outcome cannot be achieved. Most importantly, it has nothing to say about whether KM's second example (non-monotonic equilibrium) is best, or whether better long cheap talk equilibria exist. This question, that has gnawed at the authors since then, is the main question that this paper attempts to answer.

Building on Hart (1985) and Aumann and Hart (1986), AH provide a general characterization of equilibrium outcomes under cheap talk, when incomplete information is one-sided, which is as beautiful as it is intractable in a setting such as ours. Formally, it does not apply anyhow, as actions and types are assumed to take finitely many values in AH. But more importantly, it is rather unclear how one goes about applying their main characterization (Theorem B) to our problem, as solving for the di-span of the correspondence that characterizes the equilibrium payoff set of long cheap talk appears just as difficult as solving for all long cheap talk equilibria (not too surprisingly, given that their Theorem is an equivalence). Krishna's (2011) results provide additional insights into how their results specialize to the case of Sender-Receiver games, none that we would know how to take advantage of for the purpose of solving for the best equilibrium. Amitai (1996) provides a lucid account of the difficulties that extending the theory to two-sided incomplete information entails. Forges (1990) and Simon (2002) provide beautiful examples and results illustrating the need to consider infinitely-long cheap rather than finite cheap talk. Our game does not exhibit such a discontinuity, to the extent that the payoff of the best T-length straight talk equilibrium converges to the best infinite-length straight talk equilibrium. Nonetheless, it is the case that adding a round always leads to a strict improvement. Myerson (1986) provides the foundations for modeling communication in multi-stage games. Baliga and Sjöström (2004) shows how cheap talk can be usefully applied to economic environments, in particular, to arms races.

2 Straight Talk

2.1 Definitions

We revisit the basic model of cheap talk of Crawford and Sobel (1982, henceforth). The game involves a Sender and a Receiver. The Sender privately observes the state of the world θ, drawn uniformly from the unit interval. The Receiver takes an action $y \in [0,1]$, which ends the game. Realized payoffs are

$$\mathcal{U}^S(\theta, y) = -(y-\theta-b)^2, \quad \mathcal{U}^R(\theta, y) = -(y-\theta)^2,$$

for the Sender and Receiver, respectively, where $b > 0$. Players seek to maximize their expected payoff (payoff, thereafter).

We have not yet introduced an extensive form. To this end, we follow Aumann and Hart (2003, hereafter AH), and introduce a communication (or "talk") phase, which has infinitely many rounds $0, 1, 2, \ldots$ In each round, the Sender sends a message from a binary set $M = \{L, R\}$. At the end of each round, a j.c.l., which we model as a uniform draw from the unit interval, is realized. This draw is independent of θ and independent across rounds. After all messages are sent (formally, at round $\omega + 1$, where ω is the first ordinal), the receiver chooses an action, as a function of all messages and realizations of the draws. A t-period history is an element in $H_t = (M \times [0,1])^t$. A pure strategy for the Sender is a sequence of measurable maps $\sigma_1^S, \sigma_2^S, \ldots$, with $\sigma_t^S : H_{t-1} \to M$ (and H_t is endowed with the usual product structure), whereas a pure strategy for the Receiver is a measurable map $\sigma^R : H_\infty \to [0,1]$, where $H_\infty := (M \times [0,1])^\infty$ is endowed with the smallest σ-field containing all finite rectangles. See AH for further details. We note that, unlike AH, we do not need to define mixed strategies, to the extent that the Receiver's best-reply is always single-valued, and the modeling of the j.c.l. as a randomization device makes it unnecessary to introduce mixing for the Sender.[4] In addition, we have restricted attention to a binary message space rather than an arbitrary finite one. To the extent that the horizon is infinite, this is merely for convenience. However, when considering cheap-talk equilibria with two rounds only (see Section 2.2), we relax this assumption and allow for arbitrary sets of messages in each round. To simplify notation, we use closed intervals (or unions thereof) whenever describing sets of types that send a particular message (even if this implies an inconsistency regarding the strategy followed by some particular cut-off type).

The solution concept is Perfect Bayesian Equilibrium, as defined by Fudenberg and Tirole (1991, Definition 8.2).[5] We recall that, given any equilibrium, it is readily seen that payoffs of Sender and Receiver differ by a constant (b^2), so that the expression "best equilibrium" entails no ambiguity.

Note that babbling is always an equilibrium, and that it can always be specified as a continuation equilibrium after any given history h_t. If an equilibrium specifies babbling after a given history, we say that communication *ends* or *stops* in that round (after history h_t).

It is known (see GHPS) that even the mediator cannot improve on babbling when $b \geq 1/2$, and that the monotonic equilibrium constructed by KM is best among all cheap talk equilibria when $b \geq 1/8$. Hence, attention is restricted to $b \in [1/8, 1/2]$.

In this paper, we mostly focus on one particular type of equilibrium, straight talk equilibria, defined next.

Definition 1 *An equilibrium is a straight talk equilibrium if, for every $t = 0, 1, \ldots$, communication stops after one message (or both), and the unit interval can be partitioned into two sets such the continuation strategies of both Sender and Receiver are constant on each set.*

A straight talk has length $t < \infty$ if communication stops after at most t rounds (with probability 1).

The first requirement says that only one of the two messages leads to further communication, if any. The second requirement states that the uniform draw can be replaced by a Bernoulli variable whose parameter in each round can be chosen freely. See left panel of Figure 1. In this fashion, and identifying realizations of the j.c.l. that lead to the same continuation, there is only one history of length t after which communication has not stopped, a property that we will take advantage in Section 4. The *terminating message* is whichever of the two

[4] This is not to say that mixed strategies could not be defined for the Sender, whereby he randomizes over messages. Simply, we won't consider those.

[5] Their definition assumes finitely many types. The adaptation to continuous types is standard and omitted.

messages leads to communication to stop.[6]

It is worth pointing out that all equilibria investigated in the literature can be represented as straight talk equilibria. Partitional equilibria, for instance, can be represented in many ways, if we think of each interval as being identified with a particular time at which the stopping message is sent, and the j.c.l. specifies that communication continues with probability one at each round. The equilibria constructed by KM are also straight talk equilibria. But not all equilibria are straight talk equilibria, as we show below.

Given their importance in what follows, it might be useful to briefly describe the non-monotonic equilibria found by KM. Such an equilibrium is characterized by two cut-offs, $0 < x < z < 1$, and a probability $\lambda \in (0,1)$. In the first round, the Sender sends the terminating message if his type lies in $[x, z]$. If the other message is sent, the j.c.l. specifies that communication stops with probability λ. If not, a second and final message by the Sender separates the low types in $[0, x]$ from the high types in $[z, 1]$. For this to be an equilibrium, three constraints must be satisfied. First, a low type (say, type x) must prefer to say so in the second round than to claim he is high (and vice-versa). Second, types x and z must be indifferent between the two messages in the first round. This imposes two indifference conditions, pinning any two of the three variables (x, z, λ) as a function of the third. These non-linear conditions lead to an admissible solution if and only if $b \leq \frac{1}{\sqrt{8}}$. If so, there is in fact a continuum of solutions, corresponding to the value of the third variable (within some range). But only one maximizes welfare.

More intuition for this construction –and for the structure of straight-talk more generally– is given below, as well as an explanation for why $\frac{1}{\sqrt{8}}$ turns out to be critical.

2.2 Non-straight Talk Equilibria

We now show by example that equilibria exist that cannot be represented as straight talk equilibria. Our goal is not to be exhaustive, but to divide these examples into two broad categories.

The first example involves three thresholds x, y, z, with $0 < x < y < z < 1$, and two rounds of messages. The extensive form is represented in Figure 2. An initial message (here, L or R) allows the Sender's types to separate into two subsets, $S_1 := [0, x] \cup [y, z]$ on one hand, and $S_2 := [x, y] \cup [z, 1]$ on the other. If the message signals a type in S_k, communication stops (S) with probability $\lambda_k < 1$, which is determined by the j.c.l. If communication continues (C), a second round of messages leads to further revelation, with S_k being partitioned into

[6] Hopefully, no ambiguity arises when both messages are terminating.

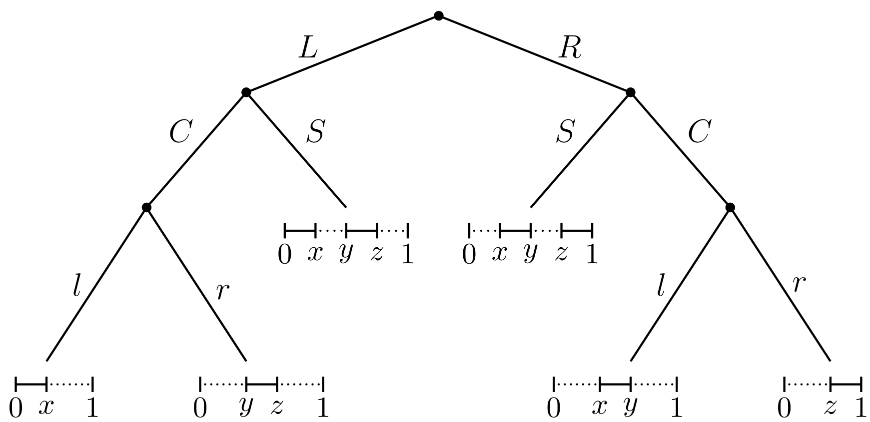

Figure 2: Non-straight talk equilibria: An overlapping example

the lower and higher interval of types (*e.g.*, $[0, x]$ vs. $[y, z]$ for S_1, and $[x, y]$ vs. $[z, 1]$ for S_2). Because $\lambda_1, \lambda_2 < 1$, this structure cannot be represented using straight talk. Because the sets S_1 and S_2 involve alternating intervals, we call such a configuration overlapping. For a concrete example of such an equilibrium, we may specify that $(x, y, z) \simeq (0.02, 0.179, 0.88)$ and $(\lambda_1, \lambda_2) \simeq (.09, 0.78)$, and $b = 0.2$.[7]

The second example, represented in Figure 3, is quite different. Departing from our convention that only two messages are available, the Sender sends one of three messages, L, M and R in the first round. The message M signals that the type is in a middle interval $[w, x]$, in which case communication stops. Message L signals types in $S_1 := [v, w] \cup [x, y]$, while message R signals types in $S_2 := [0, v] \cup [y, 1]$. If the Sender signals a type in S_k, communication stops (S) with probability $\lambda_k < 1$. If it continues (C), then a second signal allows the types in each interval to separate themselves from the types in the other remaining possible interval. Because S_1 is "inside" S_2, we refer to such a structure as nested. Again, because $\lambda_1, \lambda_2 < 1$, this structure cannot be represented using straight talk.

We note that KM's non-monotonic equilibrium is a special case in which either S_1 or S_2 is empty. It can be shown that this special case is best, in the sense that, provided one of the initial messages signals types in some set $[0, x] \cup [z, 1]$, then it is best (for efficiency) to have all other types send a common message M that stops communication.[8]

Of course, these two examples are not exhaustive. One can construct more complicated examples, possibly combining elements of both the nested and the overlapping structure.

[7] This particular equilibrium structure, involving three cut-offs x, y and z, exists if and only if $b < 1/4$.

[8] For a concrete example of such an equilibrium, we may specify that $(v, w, x, y) \simeq (0.069, 0.13, 0.90, 0.97)$, $(\lambda_1, \lambda_2) \simeq (0.99, 0.59)$, and $b = 0.26$.

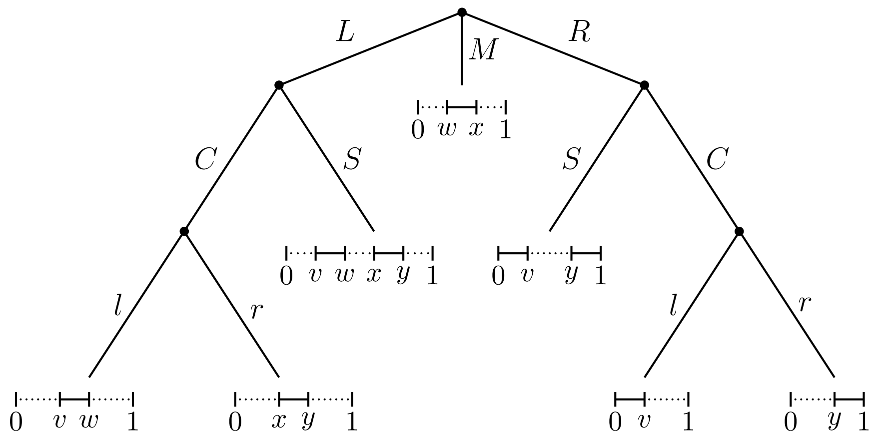

Figure 3: Non-straight talk equilibria: A nested example

Nonetheless:

Conjecture 1 *Fix $b \in [1/8, 1/\sqrt{8}]$. For every cheap talk equilibrium with two rounds of messages, there exists a two-round straight talk equilibrium (with the structure of KM's non-monotonic equilibrium) that achieves at least as high a payoff.*

This result has three implications. First, and quite remarkably, the two equilibria that KM construct turn out to be the only that may be optimal with two rounds of communication. (The monotonic one being optimal for $b < 1/8$, independent of the number of rounds.) Second, this establishes that, whether one considers the best equilibrium with one round (the partitional equilibria of CS), or the best equilibrium with two rounds (one of the two equilibria of KM), the focus on straight talk equilibrium is without loss (provided is length is unrestricted, as implementing a partitional equilibrium with straight talk, for instance, takes more than one round). Third, this result shows that the necessity to look for equilibria involving more rounds of communication is not a by-product of the focus on straight talk equilibria, but a genuine feature of the underlying incomplete information game.

2.3 Structure

KM's non-monotonic equilibrium is a straight talk equilibrium of length 2. In fact, because its specification entails a degree of freedom, there is a continuum of such equilibria. For $b > 1/4$, there is no other straight talk equilibrium with two rounds only, as the next result implies. More generally, straight talk equilibria, if they exist, must have a particular structure.

First, we focus on the case $b > 1/4$, which is somewhat simpler.

Theorem 1 *Fix $b > 1/4$. Every straight talk equilibrium has the following structure. In the first round, the terminating message is sent by all types in some middle interval $[x, z]$, $0 < x < z < 1$, so that the set of remaining types is $[0, x] \cup [z, 1]$. In all later rounds, the terminating message is sent by one and only one interval of types, such that the set of remaining types is of the kind $[y, x] \cup [z', 1]$, $z' \geq z$, $y \leq x$.*

That is, after an initial terminating message sent by a central interval of types, later terminating messages are sent by a lower interval of types that truncates one of the two intervals of types that remain. Obviously, communication may stop at any time (a special case where these intervals are empty). KM's non-monotonic equilibrium is a special case in which the second terminating message is sent by all types in $[z, 1]$ (or equivalently, all types in $[0, x]$). Later cuts to the lower or the upper interval may occur in an arbitrary order.

We now present a specific example illustrating the construction for three rounds and provide some intuition for why the later cuts must arise "from below." Here, $b = 1/4$, and $(x, z, z') = (0.078, 0.97, 0.99)$, while $(\lambda_1, \lambda_2) = (0.54, 0.006)$. The structure of this straight talk equilibrium is described in Figure 4. First, the middle interval $[x, z]$ sends a terminating message. If this message is not sent, the j.c.l. determines whether communication ends (with probability λ_1) or not. If it does not, then types in the interval $[z, z']$ send a terminating message. If this message is not sent, the j.c.l. determines whether communication ends (with probability λ_2) or not. If not, a third message reveals whether the type is in $[0, x]$ or in $[z', 1]$.

To understand why this is an equilibrium, and why in particular it is possible to introduce a second message sent by types in $[z, z']$, $z' < 1$, consider the case in which z' is very close to z, so that sending this message leads to an action approximately equal to z. Why is type z willing to send this message? Note that, for $x > z$, it holds that $\mathbf{E}[\theta \mid \theta \notin [x, z]] < z$: that is to say, conditional on communication stopping because of the j.c.l., type z would rather reveal his type. On the other hand, if communication was certain to continue ($\lambda_2 = 0$), then type z would prefer to wait and pool with higher types (it suffices that $z + b > (1 + z)/2$). Hence, there exists a value of λ_2 that makes type z willing to disclose his identity in the second round. By continuity, the same argument can be made for types z' slightly larger than z.

Given the existence of such an indifferent type z', why are the lower types within the interval those who choose to stop communication? This is because quadratic preferences exhibit decreasing absolute risk aversion: if type z' is indifferent between a "sure" action

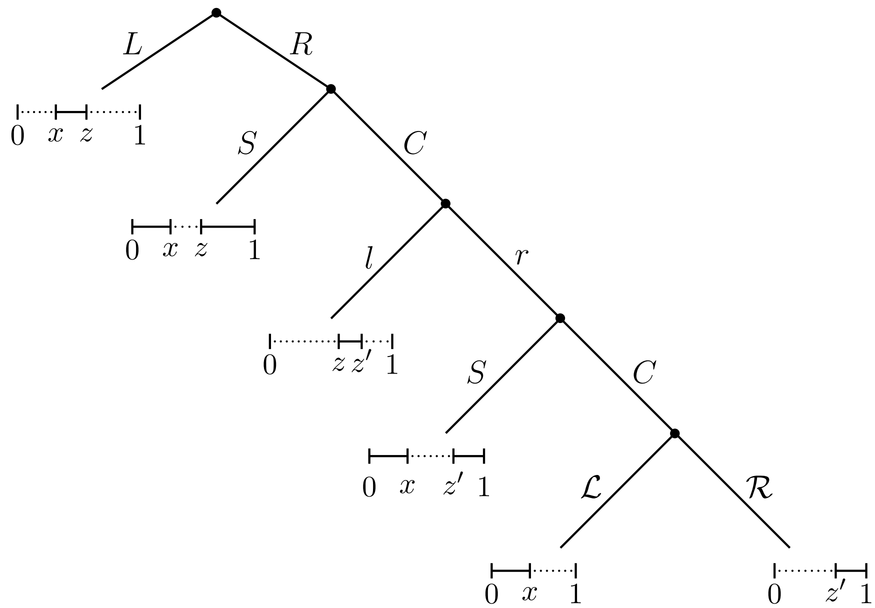

Figure 4: Straight talk equilibrium: Three rounds

$(z+z')/2$, and a lottery over $\mathbf{E}[\theta \mid \theta \notin [x, z']]$ (with probability λ_2) and $(1+z')/2$ (with complementary probability), types larger than z' prefer the lottery (we may think of a larger type, and hence a larger bliss point, as a larger wealth, since it increases the distance between this bliss point and the relevant actions, which are all lower).

This explains why such an additional cut is possible, not why it is desirable. For this, some background material is in order. As shown in GHPS, the payoff of an incentive compatible mechanism (in a cheap talk equilibrium, and more generally, under mediation) is pinned down by the payoff of the lowest type, type 0. Loosely speaking, this is because we have two sets of constraints: incentive compatibility for the Sender, and sequential rationality for the Receiver, so that, identifying an equilibrium outcome with a map from types θ to pairs $(\mathbf{E}[y(\theta)], \mathbf{Var}[y(\theta)])$, the expected action and variance that a given type obtains, we may solve for the map if the initial values (the allocation received by the lowest type) is given. Hence, it is useful to think of the objective as being to maximize the payoff of the lowest type (as opposed to the expectation of the payoff over all types). This follows from Lemma 1 of GHPS. When $b > 1/8$, it is no longer possible to give the lowest type his favorite action ($y = b$) with probability one.[9] Either type 0's allocation becomes random, or its mean grows

[9]At $b = 1/8$, the partitional equilibrium is precisely the partition $\{[0, 2b], [2b, 1]\}$, which gives type 0 his favorite action. But once $b > 1/8$, this is no longer possible, as the action for the lowest type is then above b in the partitional equilibrium. This impossibility extends to all finite cheap talk equilibria, see Theorem 3 of GHPS.

too large. Non-monotonic equilibria of KM resolve this by making the action that type 0 receives random, and specifies a conditional action $x/2$ that is too low for type 0 (relative to b). By introducing a cut at z', the payoff from type z decreases, *ceteris paribus*, as if he sends the terminating message, he gets $(z + z')/2$ rather than the larger $(z + 1)/2$. This makes pooling with the middle interval that terminates communication immediately more attractive for this type: as a result, the lowest type willing to wait rather than terminate immediately, namely z, goes up, which in turns leads to a higher action conditional on immediate termination. This pushes up the lower extremity of the middle interval, leading to a higher value of x, and consequently a higher value of $x/2$, which pushes up the payoff of type 0.

Of course, this intuition is incomplete, as it is not possible to introduce z' while keeping everything else fixed, in particular, the probabilities of termination.

Let us return to the general characterization and amend Theorem 1 for the case $b \leq 1/4$. When $b \in [1/8, 1/4]$, additional structures arise. Indeed, partitional equilibria exist, with two intervals. Plainly, such equilibria do not involve a third, middle, interval, and so cannot be covered by the previous theorem as stated. Yet, the adaptation is straightforward.

Theorem 2 *Fix $b \in (1/8, 1/4]$. Then every straight talk equilibrium has the following structure. In the first round, the terminating message is sent by either an interval $[x, 1]$, or by $[0, x]$, $x \in [0, 1]$. If the j.c.l. determines that communication continues, the structure is as in Theorem 1 for the remaining types, whether $[0, x]$ or $[x, 1]$. That is, a middle interval sends a terminating message in the second round, etc.*

3 Finite Straight Talk

3.1 Existence

In Section 2.3, we have already exhibited an example of a straight talk equilibrium with more than two rounds. As Theorem 2 hints at, more complicated straight talk equilibria can be devised. In fact, the following holds.

Theorem 3 *Fix any $b \in \left[\frac{1}{8}, \frac{1}{\sqrt{8}}\right]$, and any integer $t \geq 2$. There exists a straight talk equilibrium of length t.*

In fact, there is considerable leeway in choosing a straight talk equilibrium of a given length. Consider, for instance, an equilibrium with all further cuts arising in the interval $[z, 1]$, where

$z =: z_1$ is the upper extremity of the middle interval of types that send a terminating message in the first round. At each round, and until communication ends, an interval $[z_k, z_{k+1}]$ sends a terminating message. Call z_t the last such cut-off (that is, after $[z_{t-1}, z_t]$ sends a terminating message, and conditional on the j.c.l. specifying that communication continues, types $[0, x]$ and $[z_t, 1]$ send distinct messages). We have $t+1$ equality constraints, corresponding to the indifference of types $\theta = x, z_k,\ k = 1, \ldots, t$ between two strategies, yet we have t free variables (in addition to the cut-off values x, z_k), namely, the probabilities λ_k with which the j.c.l. specifies that communication stops in a given round. Hence, we obtain an equilibrium manifold of dimension t.[10] This is not to say that this manifold is easy to describe: it is semi-algebraic, as all equations involve polynomials in the unknown variables, but it admits no closed form solution in general. Furthermore, there are inequalities that must be satisfied corresponding to non-local incentive constraints.

Theorem 3 guarantees equilibrium existence for biases in a given range. The next theorem establishes that this range is essentially tight.

Theorem 4 *There are no (nontrivial) straight talk equilibria that end in finite time for* $b > \frac{1}{\sqrt{8}}$.

To understand why $1/\sqrt{8}$ plays such a critical role, it might be useful to consider the simplest of all straight talk equilibria, namely the non-monotonic equilibrium of KM. As the bias increases, the middle interval expands, and as x and z approach 0 and 1, it follows that

$$\mathbf{E}[\theta \in [x, z]] \to 1/2.$$

However, the conditional expectation if the j.c.l. does not terminate communicate is free, as it depends on the relative size of x vs. $1-z$, which can be chosen arbitrarily as $x \to 0, z \to 1$:

$$\mathbf{E}[\theta \notin [x, z]] \to \alpha \in [0, 1].$$

Indifference of types $x \simeq 0$ and $z \simeq 1$ then requires

$$\begin{cases} \left(\frac{1}{2} - b\right)^2 = \lambda\left(\alpha - b\right)^2 + (1-\lambda)b^2 \\ \left(\frac{1}{2} - (1+b)\right)^2 = \lambda\left(\alpha - (1+b)\right)^2 + (1-\lambda)b^2, \end{cases}$$

[10] Similarly if we choose cuts in the lower interval $[0, x]$, or in either interval depending on the round.

where λ is the probability with which communication stops because of the j.c.l. We may solve this system in two unknowns, namely λ and α, which gives

$$\lambda = \frac{4b^2}{1-4b^2}, \text{ and } \alpha = \frac{1}{2} - \frac{1}{4b} + 2b,$$

and so λ is less than one only if

$$b \leq \frac{1}{\sqrt{8}}.$$

3.2 Welfare

Not all straight talk equilibria are equally desirable. Welfare maximization allows us to further restrict the structure of straight talk equilibria that are candidates for optimality.

Conjecture 2 *Fix any $b \in \left[\frac{1}{8}, \frac{1}{\sqrt{8}}\right]$, and any integer $t \geq 2$. The best straight talk equilibrium of length t does not involve any cuts in the interval $[0, x]$, where $[x, z]$ $(0 < x < z)$ denotes the set of types that send the terminating message in the first round.*

That is to say, all types in the interval $[0, x]$ use the same strategy. While we did not insist on existence when stating Theorem 2, it is worth mentioning that the possibility of cuts in the interval $[0, x]$, as described in this theorem, is not simply a mere possibility that we were not able to rule out. Equilibria involving such cuts do exist, but they are dominated by equilibria that do not involve such cuts. (In fact, they are dominated by the best non-monotonic equilibrium of KM.)

Next, we show that adding rounds increases welfare.

Theorem 5 *Fix any $b \in \left[\frac{1}{8}, \frac{1}{\sqrt{8}}\right]$, and any integer $t \geq 2$. The best straight talk equilibrium of length $t+1$ yields a higher payoff than all straight talk equilibria of length no larger than t.*

Considering that longer talk phases help, it is natural to ask how the structure of the best equilibrium of a given length changes as rounds of cheap talk are added. As Figure 5 illustrates for the case of 11 rounds, the additional cuts are distributed relatively evenly in the interval $[z_1, 1]$ (of course, z_1 is itself a function of the length). Here, each additional cut is indicated by the threshold z_k, and the action in case the j.c.l. ends communication in a given round by y_k. As is clear, the sequence of y_k's is decreasing, eventually dropping below x itself, but remaining above $x/2$, the action taken if the j.c.l. never stops the communication and the Sender's type is in $[0, x]$. What is not represented here are the probabilities that

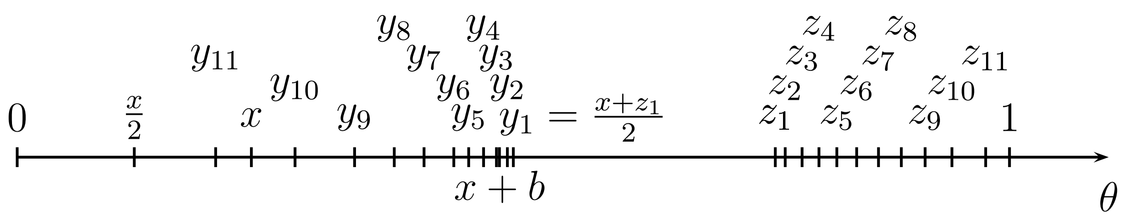

Figure 5: Best straight talk equilibrium of length 11 ($b = 1/4$)

communication stops because of the j.c.l., which decrease as the number of rounds increases, with a cumulative probability that converges, so that the probability that types in $[0, x]$ get to signal that they belong to this interval is neither zero nor one. Yet, describing the sequence formally is difficult, and to describe its limit, it is best to consider an auxiliary game in continuous time, defined next.

4 An Auxiliary Stopping Game

4.1 Definition

One of the benefits of straight talk equilibria is that, because there is only possible history of talk of length t after which communication has not stopped, it can be represented as a stopping game. Hence, it can be modeled in continuous time, circumvented the difficulties associated with talking limits in discrete time. (See Theorem 3 below for convergence.)

The game is played in continuous time, indexed by $t \in T := [0, 1] \cup \{\infty\}$. The addition of "$\infty$" simply captures the fact that the Receiver might act after the entire talk phase, which lasts potentially up to time $t = 1$ (in particular, it does last up to $t = 1$ with positive probability in the equilibria we are interested in). Hence, we might as well think of this time as $t = 2$, or $t =$ "1+".

A strategy for the Sender is a measurable map $t : [0, 1] \to T$, with the interpretation of $t(\theta)$ as the time at which the sender of type θ stops the clock. Given a strategy $t(\cdot)$, define the (right-continuous) c.d.f. $G : T \to [0, 1]$ by, for all t, $G(t) = \Pr[\theta : t(\theta) \leq t]$.

Further, let be given an arbitrary c.d.f. $F : [0, 1] \to [0, 1]$, and a random variable τ_F on $[0, 1]$ (defined on an arbitrary probability space), independent of θ, and with distribution function F. The interpretation of τ_F is that of a random time at which the jointly controlled lottery calls for communication to stop, in case the sender hasn't stopped the clock already. The distribution F is part of the equilibrium specification, but equilibrium imposes no restriction on it.

Finally, let be given two maps $y_F, y_G : [0,1] \to \mathbf{R}$. The expected cost for the sender of type θ from choosing time $t \leq 1$ is given by

$$U(t \mid \theta) = \int_{[0,t)} (y_F(s) - (\theta + b))^2 \mathrm{d}F(s) + (1 - F(t_-))(y_G(t) - (\theta + b))^2, \tag{1}$$

where $F(t_-)$ is the left-limit of F at t ($F(0_-) := 0$ and the integral is also zero for $t = 0$); choosing $t = \infty$ yields

$$U(\infty \mid \theta) = \int_{[0,1]} (y_F(s) - (\theta + b))^2 \mathrm{d}F(s). \tag{2}$$

Definition 2 *An equilibrium is a distribution F, a strategy $t(\cdot)$, and two maps y_F, y_G : $[0,1] \to \mathbf{R}$ such that:*

1. *It holds that*
$$y_F(t) = \mathbf{E}\left[\theta \mid t(\theta) > t\right], \qquad y_G(t) = \mathbf{E}\left[\theta \mid t(\theta) = t\right], \tag{3}$$
whenever these regular conditional probabilities are well-defined (set y_F and y_G arbitrarily if $t >$ supp G);

2. *For all θ, $t(\theta)$ minimizes $U(t \mid \theta)$ (as defined in (1)–(2), given y_F, y_G) over $t \in T$.*

Plainly, communication is restricted in a particular way here: a sender chooses when to say "stop," if ever, and the sender has to act at either the time the sender says so, or the j.c.l. calls for babbling, whichever comes first. Yet, this communication protocol is rich enough to capture the limiting outcome of straight talk equilibrium of length t, as $t \to \infty$. We note that (3) defines the receiver's actions uniquely only on path, but the specification of off-path actions is irrelevant for the distribution over outcomes (actions and types): "silent" periods can be removed, and active periods stitched together by simply changing variables in the strategy.

As a side remark, continuous time brings out the close connection between long cheap talk and a repeated cheap talk game in continuous time between a patient sender with permanent type (who does not discount future) and a myopic receiver who best responds to her beliefs at every instant in time, as in, for instance, Golosov *et al.* (2014). To see this equivalence, take F in the definition of the continuous stopping game to be uniform, so that each time instant is given equal weight in the payoff of the receiver:

$$U(t \mid \theta) = \int_0^t (y_F(s) - \theta - b)^2 \, \mathrm{d}s + \int_t^1 (y_G(t) - \theta - b)^2 \, \mathrm{d}s.$$

More generally, the distribution F can be mapped to a (time-varying) discount rate.[11]

4.2 Existence and Optimality

We now define a particular class of equilibria. To build some intuition for the distributions involved, suppose that cuts (after the initial interval is ruled out at the first instant) occurs in intervals of small length $\varepsilon > 0$. Specifically, at any time, if communication is not over after t rounds, then the sender's types in $[z, z+\varepsilon]$ send a terminating message, where z is the lowest type left in the upper interval; if this message is not sent, with probability $m(z)\varepsilon$, the j.c.l. specifies termination and the receiver takes y, his best estimate of the state given that all states in $[0, x] \cup [z+\varepsilon, 1]$ are equally likely. If the j.c.l. specifies that communication continues, types in $[z+\varepsilon, z+2\varepsilon]$ are next, etc. Type $z+\varepsilon$ must be indifferent between separating with $[z, z+\varepsilon]$ or separating in the following round, assuming he gets the chance to do so. Hence we must have, for all z,

$$\begin{aligned}\left(z + \frac{\varepsilon}{2} - (z+\varepsilon+b)\right)^2 \quad &= \quad m(z)\varepsilon \left(y^P - (z+\varepsilon+b)\right)^2 \\ &\quad + (1 - m(z)\varepsilon)\left(z + 3\frac{\varepsilon}{2} - (z+\varepsilon+b)\right)^2,\end{aligned}$$

where

$$y = \frac{x}{x+1-z-\varepsilon}\frac{x}{2} + \frac{1-z-\varepsilon}{x+1-z-\varepsilon}\frac{z+\varepsilon+1}{2}.$$

Taking limits ($\varepsilon \to 0$) gives

$$m(z) = \frac{8b}{\frac{(1+(x-z)^2 - 2b(1+x-z) - 2z)^2}{(1+x-z)^2} - 4b^2}.$$

This is the hazard rate at which communication ends because of the j.c.l., pinned down by the Sender's indifference. Computing $e^{-\int m(z)\mathrm{d}z}$ yields a c.d.f. up to a constant, which is

[11] An equilibrium as defined below in Section 4.2 translates as follows: after signaling at $t = 0$, there is initial pooling until $t = M$; separation occurs during times in $\left(M, 1 - \frac{(1-M)H(1)}{H(1-ax)}\right)$, with type $\theta \in [1-ax, 1]$ separating at $t(\theta) = 1 - \frac{(1-M)H(\theta)}{H(1-ax)}$; the final payoff accumulation take place from $t = 1 - \frac{(1-M)H(1)}{H(1-ax)}$ to $t = 1$.

introduced next. Namely, for $x \in [0, 1/2]$ and $\theta \in [1-x, 1]$, define

$$H(\theta) := \left(1 - \frac{4b(1+x-\theta)}{(x-\theta)^2 + 1 - 2\theta}\right) \left(\frac{1 + \frac{2b+\theta-x-1}{\sqrt{2}\sqrt{2b^2+x}}}{1 - \frac{2b+\theta-x-1}{\sqrt{2}\sqrt{2b^2+x}}}\right)^{\frac{b}{\sqrt{b^2+\frac{x}{2}}}}.$$

The constant pinning down the c.d.f. depends on the size of the middle interval that sends the initial termination message, call it $[x, z]$. It is often more convenient to work with a than z, where $z = 1 - ax$ ($a < 1$ because $1 - z < x$ in straight talk equilibria). To define the missing constant, let, for $a \in [0, 1]$,

$$M := \frac{(a+1)^2((a+1)x-1)((a+1)x-1-4b)}{((a+1)^2x-2)\left((a+1)^2x-2-4(a+1)b\right)},$$

which is in $[0, 1]$ if $(1+a)x \leq 1-2b$, a condition that is necessary for existence. To summarize, we are interested in pairs (x, a) in $R := \{(x, a) : 0 < (1+a)x < 1-2b, 0 < a < 1\}$.

Given $b \in \left[\frac{1}{8}, \frac{1}{\sqrt{8}}\right]$, we may now define a distribution F and a strategy $t(\cdot)$ by a pair $(x, a) \in R$, as follows.

1. The strategy t is given by

$$t(\theta) = 0 \ \forall \theta \in [x, 1-ax], \tag{4}$$

$$t(\theta) = \theta \ \forall \theta > 1-ax, \tag{5}$$

$$t(\theta) = \infty \ \forall \theta < x. \tag{6}$$

2. The distribution F is given by

$$F(0) = F(1-ax) = M, \tag{7}$$

$$F(\theta) = 1 - (1-M)\frac{H(\theta)}{H(1-ax)} \ \forall \theta \in (1-ax, 1). \tag{8}$$

By definition of F, all types in $(1-ax, 1)$ follow optimal strategies. An additional restriction on the pair arises from the indifference of type x (between sending the terminating message in the first round and not). With some algebra, this indifference translates into:

$$\int_{1-ax}^{1} \frac{H(s)\left(s^2-2s(x+1)+x^2+1\right)\left(2b(x+1-s)+(s-x)^2+2x-1\right)}{2(1+x-s)^3} \mathrm{d}s$$
$$= \frac{(1-a)\left(4ab+2(a+1)^2x-3a+4b-1\right)H(1-ax)}{4(1-M)(a+1)^2}. \tag{9}$$

Note that the pair (x, a) pins down a unique candidate F, G via (4)–(8). The following theorem formalizes our construction.

Theorem 6 *Every pair $(x, a) \in R$ such that (9) holds defines an equilibrium via (4)–(8).*

A natural question, then, is whether (4)–(8) can be satisfied. Given x, what values of a, in any, are consistent with a solution? This depends upon wether b is above $1/4$ or not.

Theorem 7 *For each $b \in \left[\frac{1}{4}, \frac{1}{\sqrt{8}}\right]$, there exists $\bar{x} > 0$, and an increasing map $[0, \bar{x}] \mapsto a(x) \in \left[1 - \frac{8b^2 - 1}{4\left(b - \frac{1}{2}\right)\left(b + \frac{1}{4}\right)}, 1\right]$, such that a solution to (4)–(9) exists if and only if $x < \bar{x}$ and $a = a(x)$. There exists no solution for $b > 1/\sqrt{8}$.*

We now turn to the more difficult case in which $b < 1/4$. In that case, it is easier to express x in terms of a.

Theorem 8 *For each $b \in [1/8, 1/4]$, and all $a \in [0, 1]$, there exists a unique $x(a) \in (0, 1/2)$ such that $(a, x(a))$ defines an equilibrium.*

However, each x need not map into a unique a: indeed, for $b \leq \hat{b} \approx 1/6$, $x(a)$ admits an interior maximum in a, so that for every $x \in [x(1), \max_a x(a))$, there exists two values of a with $x(a) = x$. It is readily verified that $x(0) = \frac{1}{2}(1 - 4b)$. Hence, for $b \in [1/8, 1/4]$, x varies from $\frac{1}{2}(1 - 4b)$ and some upper bound $\max_a x(a)$, while for $b \geq 1/4$, x varies from 0 to $\max_a x(a) = x(1)$. (There appears to be no simple formula for these maxima.)

Figure 6 displays the pairs (a, x) that satisfy the constraint (9) (blue curve), as well as the tangent isocost curve given by (10) for two levels of b (the right panel shows how two values of a are consistent with the same x for low b; the right panel illustrates that for high enough b, pairs satisfying the constraint only exist if a is high enough). Here, a is in abcissa, x is in ordinate. As is clear, there is a unique optimal choice of (a, x), and the optimal a is increasing in b, at least comparing these two values – Figure 7 elucidates that this is true more generally.

Given an equilibrium (F, t, y_F, y_G) defined by a pair (x, a), let $U(a, x) = \mathbf{E}_\theta[U(t(\theta), \theta)]$ denote the resulting cost (for the sender). This cost can be "explicitly" solved for, namely,[12]

$$\begin{aligned} U(a,x) &= b^2 - \frac{1}{12} + \frac{(a-1)^2 x((a+1)x - 1)}{4(a+1)} \\ &\quad - \frac{1-M}{H(1-ax)} \int_{1-ax}^{1} \frac{H(s)\left((x+1-s)^2 - 2x\right)^2}{4(x+1-s)^2} \mathrm{d}s. \end{aligned} \tag{10}$$

[12]The integrals appearing here and in (9) can be solved in terms of hypergeometric functions, but doing so provides no additional insight.

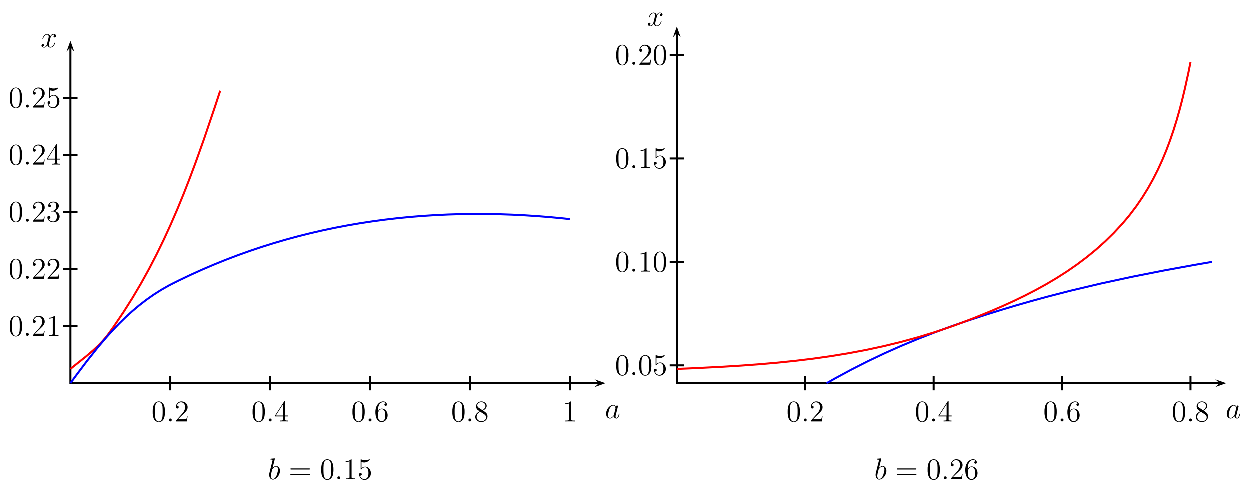

Figure 6: Equilibrium constraint on the pair (a, x) (blue curve) and isocost curve (red curve).

Hence, solving for the best equilibrium is as simple as maximizing (10) over $(x, a) \in R$ subject to (9), a task that is straightforward —numerically.

Figure 7 displays the optimal choice of x and z as a function of $b \in [1/8, 1/\sqrt{8}]$. As is clear, the optimal value of x is decreasing in b: the higher the bias, the less "room" for a non-monotonic equilibrium. In contrast, the optimal value of z is not monotone. It vanishes at 1 whether b equals $1/8$ or $1/\sqrt{8}$.

4.3 Convergence

What is the relationship between our continuous-time game and our original discrete-time cheap talk game?

Note that an equilibrium parametrized by (x, a) induces a distribution on pairs (θ, y). Write $\mu(a, x) \in \Delta([0, 1] \times \mathbf{R})$ for this distribution. Similarly, a given straight talk equilibrium of length t induces a distribution on this set, call it μ_t. We let μ_t^* denote this distribution for a straight talk equilibrium of length t that is optimal within the class of straight talk equilibria of length no more than t.

First, our continuous-time equilibria are limits of discrete-time equilibria.

Theorem 9 *Given an equilibrium in the continuous-time game, with distribution $\mu(a, x)$, there exists a sequence of straight talk equilibria, indexed by their length, with distribution μ_t, such that μ_t (weakly) converges to $\mu(a, x)$ as $t \to \infty$.*

Second, our continuous-time game (and the class of equilibria considered) is rich enough to capture the best straight talk equilibrium, in the following sense.

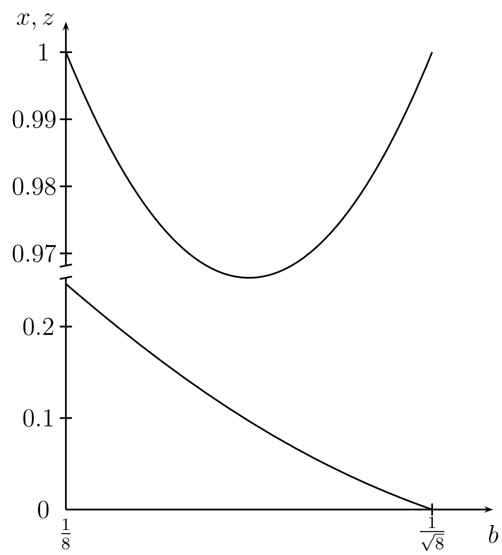

Figure 7: The optimal values of x and z as a function of b.

Conjecture 3 *It holds that μ_t^* (weakly) converges to $\mu(a, x)$ as $t \to \infty$, where (x, a) defines the best equilibrium (that is, it minimizes $U(a, x)$ over all equilibria defined by pairs (x, a)).*

Finally, it is natural to ask how much an improvement longer talk phases yield. As Figure 8 shows, the improvement from adding one more round of talk decreases very fast. The dashed line shows how much the payoff improves when adding one round to the non-monotonic KM equilibrium, while the solid line looks at the payoff improvement from considering infinite straight talk rather than the non-monotonic KM equilibrium.

5 Concluding Comments

It is our hope that the results and methods of this paper stimulate economic research on long cheap talk. Long cheap talk does matter, and for simple classes of equilibria, it can be modeled as a rather standard continuous-time stopping game. Yet, as we have shown, within the framework of CS, there is no discontinuity in the payoff as the length of the horizon increases, unlike in Forges (1990)'s remarkable example. On the other hand, unlike in her example, communication does not end almost surely in finite time.[13]

Perhaps surprising is the finding that the number of equilibrium messages (or possible

[13]It is a "folk" but open conjecture that, without loss for the equilibrium payoff characterization, long cheap ends in finite time a.s. if the Bayesian game is finite. Plainly, "slicing" the state space as happens in the best equilibrium relies on the infinite state space of CS.

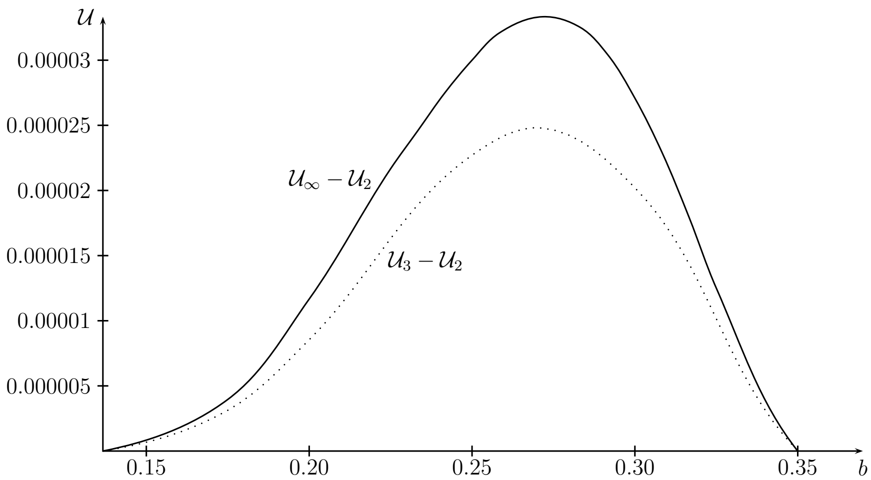

Figure 8: Payoff difference between best straight talk equilibrium of length 2 and 3 (dashed), vs. the difference between best straight talk equilibrium of length 2 and infinity (solid).

actions) is not decreasing in the bias. More aligned interests means a higher equilibrium payoff, but it does not necessarily imply richer communication. In our view, understanding more broadly in which games communication is helpful is perhaps the most important open question in this literature.

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Appendix A. Proofs for Section 2

Proof of Theorem 1. The proposition takes four steps to complete. Lemma 1 through 4 are these four steps.

Lemma 1 *At any stage, the exiting types must be a convex set, namely an interval.*

Proof. For a sequence of actions, let y and s^2 be its mean and variance. Now consider two sequences with y_1, s_1^2 and y_2, s_2^2. If types θ and θ' both prefer sequence 1 to sequence 2, then

$$-y_1^2 - s_1^2 + 2(\theta + b)y_1 - (\theta + b)^2 \geq -y_2^2 - s_2^2 + 2(\theta + b)y_2 - (\theta + b)^2$$
$$-y_1^2 - s_1^2 + 2(\theta' + b)y_1 - (\theta' + b)^2 \geq -y_2^2 - s_2^2 + 2(\theta' + b)y_2 - (\theta' + b)^2$$

so that

$$\begin{aligned} & -y_1^2 - s_1^2 + 2(\lambda\theta + (1-\lambda)\theta' + b)y_1 - (\lambda\theta + (1-\lambda)\theta' + b)^2 \\ \geq \;& -y_2^2 - s_2^2 + 2(\lambda\theta + (1-\lambda)\theta' + b)y_2 - (\lambda\theta + (1-\lambda)\theta' + b)^2 \end{aligned}$$

for all $\lambda \in [0,1]$.

So, if exiting at some stage t is best for types θ and θ', then it is also best for types in between. ■

Lemma 2 *The first exiting types must consist an interval $I = [d,u]$ with $0 < d < u < 1$ and $u - d \geq 2b$.*

Proof. By Lemma 1, the first exiting set must be an interval. It can be either corner or interior.

Suppose it is in the lower corner, i.e. $d = 0$. Then type u is indifferent between action $\frac{u}{2}$ and the continuation payoff in $[u,1]$. Since the continuation talk is a subgame, the sender can guarantee himself at least the babbling payoff in that subgame, which is $-(\frac{u+1}{2} - (u+b))^2$. Therefore

$$\begin{aligned} & -(\frac{u}{2} - (u+b))^2 \geq -(\frac{u+1}{2} - (u+b))^2 \\ \Rightarrow \quad & 2b \leq \frac{1}{2} - u < \frac{1}{2} \\ \Rightarrow \quad & b < \frac{1}{4} \end{aligned}$$

which is a contradiction.

Suppose instead that it is in the upper corner, i.e. $u = 1$. Then type d is indifferent between action $\frac{d+1}{2}$ and the continuation payoff in $[0,d]$. Since the continuation talk is a subgame, the best result for the sender is the sure action at d, which gives $-b^2$. Therefore

$$\begin{aligned} & -(\frac{d+1}{2} - (d+b))^2 \leq -b^2 \\ \Rightarrow \quad & 2b \leq \frac{1}{2} - \frac{d}{2} < \frac{1}{2} \\ \Rightarrow \quad & b < \frac{1}{4} \end{aligned}$$

which is also a contradiction.

So, the interval must be interior. Let a_0 be the corresponding action. From sequential rationality we know that some type $\theta \geq a_0$ must belong to the exiting set. If $a_0 \leq b$, then a_0 is the bliss point for the fictitious type $a_0 - b < 0$, who must also exit first. By the argument in Lemma 1, all types in $[0, a_0]$ must exit, contradicting to being an interior interval.

So we must have $a_0 > b$, and $[a_0 - b, a_0]$ must exit. Furthermore, by sequential rationality and Lemma 1, $[a_0, a_0 + b]$ must also exit. So $u - d \geq 2b$. ■

Lemma 3 $d + u > 1$.

Proof. Suppose instead that $d + u \leq 1$. Let $D = [0, d]$ and $U = [u, 1]$. According to Lemma 1 and Lemma 2, a type in D and a type in U cannot exit simultaneously. So, after Stage 0 when $[d, u]$ exits, the next exiting types must be either in U or D.

If they are in U, then fix a type θ within this set. Note that θ gets a lottery between $y_0 = \frac{d^2+1-u^2}{2(d+1-u)}$ and some action $a_1 \in U$. First, $a_1 \geq u > a_0$, and $a_1 - (u+b) \leq 1 - (u+b) <\leq 1 - (a_0 + 2b) < 2b - a_0 < u + b - a_0$, so that a_1 is closer than a_0 to $u + b$. For u not to mimic θ, it must be that y_0 is farther than a_0 from $u + b$. Since $d + u \leq 1$, we know $y_0 \geq a_0$, so $y_0 > 2(u + b) - a_0 \geq u + 3b > 4b > 1$, which is impossible.

If they are in D, then call the supremum of such types θ. It gets a lottery between y_0 and some action $a_1 \in D$, and it should weakly prefer exiting at Stage 1 to exiting at Stage 0 (mimicking type d). Since $y_0 \geq a_0 > \theta + b$, we must have $a_0 - (\theta + b) > \theta + b - a_1$. By sequential rationality, $a_1 \leq \theta$, so that $2b < a_0 - \theta < a_0 \leq \frac{1}{2}$. This means $b < 1/4$, a contradiction of our assumption.

In sum, it must be the case that $d + u > 1$. ■

Lemma 4 *Within set U or within set D, the exit time is non-decreasing in type.*

Proof. First we consider the types in the top section U. If all types in U exit at the same time, then the claim is satisfied trivially. If there are different exiting times, then by continuity in utility, there must be an indifferent type θ dividing two action sequences on the left and right. Suppose both sides exit at Stages t_1 and t_2 with $t_1 < t_2$, then conditional on reaching t_1, the continuation sequences of actions on both sides becomes a sure action y_1 versus a lottery with mean y_2 and variance $s_2^2 > 0$. Indifference condition reads

$$\begin{aligned} & -y_1^2 + 2(\theta + b)y_1 - (\theta + b)^2 = -y_2^2 - s_2^2 + 2(\theta + b)y_2 - (\theta + b)^2 \\ \Leftrightarrow \quad & -y_1^2 + 2(\theta' + b)y_1 = -y_2^2 - s_2^2 + 2(\theta' + b)y_2 \end{aligned}$$

so that

$$-y_1^2 + 2(\theta' + b)y_1 < -y_2^2 + 2(\theta' + b)y_2$$

Since $b > \frac{1}{4}$, we must have $\theta > u > \frac{1}{2} > 1 - 2b$ by virtue of Lemma 3. So $\theta + b > \frac{1+\theta}{2}$. By sequential rationality and Lemma 1, $y_1 \leq \frac{1+\theta}{2} < \theta + b$ because the interval that has the highest expected value and contains θ is $[\theta, 1]$. We know that on $[0, \theta + b]$, the function $f(w) = -w^2 + 2(\theta + b)w$ is increasing. Hence $y_1 < y_2$.

If we take the derivative of $(-y_1^2 + 2(\theta' + b)y_1) - (-y_2^2 + 2(\theta' + b)y_2)$ with respect to θ, we get $2(y_1 - y_2) < 0$, meaning higher types will prefer exiting later.

Now we consider the types in the bottom section D. By the same logic we obtain the inequality $-y_1^2+2(\theta'+b)y_1 < -y_2^2+2(\theta'+b)y_2$. By Lemma 1, we know $d \leq u-2b < 1-2b < 2b$. With that, we have $\theta + b > \frac{d+\theta}{2}$. Similarly, $y_1 \leq \frac{d+\theta}{2} < \theta + b$ because the interval $[\theta, d]$ has the highest expected value that contains θ, by Lemma 1. So again, $y_1 < y_2$. The first order condition for the difference of utilities gives $2(y_1 - y_2) < 0$, therefore higher types in the bottom section also exit later. ■

This completes the proof of Theorem 1. ■

Appendix B. Proofs for Section 3

Proof of Theorem 3. KM^T equilibrium ($T \geq 2$) is defined by $(x, z_1, \ldots, z_T, p_1, \ldots, p_T)$ where $0 < x < z_1 < \ldots < z_T < 1$ and $p_1, \ldots, p_T \in (0, 1)$.

Denote $y_i = \frac{x}{x+1-z_i}\frac{x}{2} + \frac{1-z_i}{x+1-z_i}\frac{1+z_i}{2}$. There are $T+1$ equilibrium conditions:

$$\begin{cases} \left(\frac{x+z_1}{2} - x - b\right)^2 = p_1\left(y_1 - x - b\right)^2 + (1-p_1)\, p_2\left(y_2 - x - b\right)^2 + ... \\ \qquad\qquad\qquad\qquad + (1-p_1)\,...\,(1-p_{T-1})\, p_T\left(y_T - x - b\right)^2 \\ \qquad\qquad\qquad + (1-p_1)\,...\,(1-p_{T-1})\,(1-p_T)\left(\frac{x}{2} - x - b\right)^2 \\ \left(\frac{x+z_1}{2} - z_1 - b\right)^2 = p_1\left(y_1 - z_1 - b\right)^2 + (1-p_1)\left(\frac{z_1+z_2}{2} - z_1 - b\right)^2 \\ \left(\frac{z_1+z_2}{2} - z_2 - b\right)^2 = p_2\left(y_2 - z_2 - b\right)^2 + (1-p_2)\left(\frac{z_2+z_3}{2} - z_2 - b\right)^2 \\ ... = ... \\ \left(\frac{z_{T-1}+z_T}{2} - z_T - b\right)^2 = p_T\left(y_T - z_T - b\right)^2 + (1-p_T)\left(\frac{z_T+1}{2} - z_T - b\right)^2 \end{cases}$$

The expected payoff of the lowest type, i.e. the "welfare", in KM^T equilibrium is

$$\begin{aligned} &-p_1\left(y_1 - b\right)^2 - (1-p_1)\, p_2\left(y_2 - b\right)^2 + ... \\ &\qquad - (1-p_1)\,...\,(1-p_{T-1})\, p_T\left(y_T - b\right)^2 \\ &- (1-p_1)\,...\,(1-p_{T-1})\,(1-p_T)\left(\frac{x}{2} - b\right)^2 \end{aligned}$$

KM^{T+1} equilibrium $(x, z_1, ..., z_{T+1}, p_1, ..., p_{T+1})$ has the same equations 2 through T, equation 1 has to be modified to

$$\begin{aligned}\left(\tfrac{x+z_1}{2}-x-b\right)^2 = p_1\left(y_1-x-b\right)^2+\left(1-p_1\right)p_2\left(y_2-x-b\right)^2+...\\ +\left(1-p_1\right)...\left(1-p_{T-1}\right)p_T\left(y_T-x-b\right)^2\\ +\left(1-p_1\right)...\left(1-p_{T-1}\right)\left(1-p_T\right)p_{T+1}\left(y_{T+1}-x-b\right)^2\\ +\left(1-p_1\right)...\left(1-p_{T-1}\right)\left(1-p_T\right)\left(1-p_{T+1}\right)\left(\tfrac{x}{2}-x-b\right)^2\end{aligned}\tag{11}$$

equation $T+1$ has to be modified to

$$\left(\frac{z_{T-1}+z_T}{2}-z_T-b\right)^2 = p_T\left(y_T-z_T-b\right)^2+\left(1-p_T\right)\left(\frac{z_T+z_{T+1}}{2}-z_T-b\right)^2$$

and there is a new equation $T+2$:

$$\left(\frac{z_T+z_{T+1}}{2}-z_{T+1}-b\right)^2 = p_{T+1}\left(y_{T+1}-z_{T+1}-b\right)^2+\left(1-p_{T+1}\right)\left(\frac{z_{T+1}+1}{2}-z_{T+1}-b\right)^2$$

Note that using all equations except the first one we can solve for p_i for $i=1,...,T+1$:

$$p_i=\frac{\left(z_{i+1}-z_{i-1}\right)\left(4b+2z_i-z_{i-1}-z_{i+1}\right)}{\left(z_i+z_{i+1}-2y_i\right)\left(4b+3z_i-2y_i-z_{i+1}\right)}\tag{12}$$

where we take $z_0:=x$ and $z_{T+2}:=1$. Thus p_i is a function of $(x, z_{i-1}, z_i, z_{i+1})$.

Thus KM^{T+1} equilibrium can be defined by $(x, z_1, ..., z_{T+1})$ subject to (i) $p_i\in(0,1)$ for $i=1,...,T+1$ given by (12); (ii) equation (11) (with p_i substituted in from (12)). The remainder of the proof is completed by two lemmas below.

Lemma 5 *Let $b>\frac{1}{8}$. In any KM^T equilibrium: $x>1-z_1$.*

Proof. Suppose there is KM^T equilibrium such that $x<1-z_1$.

Recall that incentive constraint for type x implies $x+b\le\frac{x+z_1}{2}$, which can be rewritten as

$$z_1-x\ge 2b\tag{13}$$

Type z_1 is indifferent between $\frac{x+z_1}{2}$ and a lottery with outcomes y_1 and $\frac{z_1+z_2}{2}$. Note that $\frac{z_1+z_2}{2}\in\left(\frac{x+z_1}{2},\frac{1+z_1}{2}\right)$. Also note that $y_1\le\frac{1+z_1}{2}$, and $x<1-z_1$ implies that

$$y_1-\frac{x+z_1}{2}=\frac{1-z_1-x}{2\left(x+1-z_1\right)}>0$$

Hence we also have $y_1\in\left(\frac{x+z_1}{2},\frac{1+z_1}{2}\right)$.

The ideal point of type z_1 cannot be closer to $\frac{1+z_1}{2}$ than to $\frac{x+z_1}{2}$, because that would imply that type z_1 strictly prefers lottery over y_1 and $\frac{z_1+z_2}{2}$ to action $\frac{x+z_1}{2}$. Hence $z_1 + b \leq \frac{1}{2}\left(\frac{x+z_1}{2} + \frac{1+z_1}{2}\right)$ which can be rewritten as

$$1 - z_1 \geq z_1 - x + 4b \tag{14}$$

Combine (13) and (14):

$$1 \geq (1 - z_1) + (z_1 - x) \geq (z_1 - x + 4b) + 2b \geq 8b$$

which contradicts $b > \frac{1}{8}$.

Now suppose there is KM^T equilibrium such that $x = 1 - z_1$. Then $\frac{x+z_1}{2} = y_1 = \frac{1}{2}$. Thus from (12) we have $p_1 = 1$. So this is not KM^T equilibrium. ■

Lemma 6 *Let $b > \frac{1}{8}$ and suppose there exists KM^T equilibrium, where $T \geq 2$, defined by $(\overline{x}, \overline{z}_1, ..., \overline{z}_T)$ where $0 < \overline{x} < \overline{z}_1 < ... < \overline{z}_T < 1$. Then there exists KM^{T+1} equilibrium $(x, z_1, ..., z_{T+1})$ with $x = \overline{x}$, $z_i = \overline{z}_i$ for $i = 1, ..., T-1$, and z_T, z_{T+1} that satisfy $\overline{z}_{T-1} < z_T < z_{T+1} < 1$.*

Proof. Note that y_i for $i = 1, ..., T-1$ are the same in KM^T equilibrium, so we'll denote them as $\overline{y}_i$. Since p_i is a function of $(x, z_{i-1}, z_i, z_{i+1})$, we will have $p_i = \overline{p}_i$ for $i = 1, ..., T-2$, and p_{T-1}, p_T, p_{T+1} will be determined according to (12).

First we will assume $\overline{z}_{T-1} < z_T < z_{T+1} < 1$ and establish a few useful properties. Then we will show that $\overline{z}_{T-1} < z_T < z_{T+1} < 1$ implies $p_{T-1}, p_T, p_{T+1} \in (0,1)$. After that we will show that there exist such z_T, z_{T+1} that satisfy (11).

Denote $a_i = \frac{1-z_i}{x}$ for $i = 1, ..., T+1$, $a_0 = \frac{1-x}{x}$ and $a_{T+2} = 0$. By Claim 1: $1 > a_1 > ... > a_{T+1} > 0$. To indicate that $\overline{x}, \overline{z}_1, ..., \overline{z}_{T-1}$ are fixed we will write $\overline{a}_0, \overline{a}_1, ..., \overline{a}_{T-1}$.

Note that (13) can be written as $1 \geq 2b + \overline{x}\left(\overline{a}_1 + 1\right)$, and this implies

$$1 \geq 2b + \overline{x}\left(a_i + 1\right) \text{ for every } i = 1, ..., T+1 \tag{15}$$

Using (15) and the fact that $a_i < 1$ for every $i = 1, ..., T+1$ we get

$$2 \geq 4b + 2\overline{x}\left(a_i + 1\right) > \overline{x}\left(a_i + 1\right)\left(a_{i-1} + 1\right)$$

Also note that $a_0 = \frac{1-x}{x}$ and $a_1 < 1$ imply

$$\overline{x}\left(a_1+1\right)\left(a_0+1\right) = a_1 + 1 < 2$$

Hence

$$2 - \overline{x}\left(a_i+1\right)\left(a_{i-1}+1\right) > 0 \text{ for every } i = 1, ..., T+1 \tag{16}$$

Next note that

$$8b > 1 = \overline{x} + \left(\overline{z}_1 - x\right) + \left(1 - \overline{z}_1\right) \geq \overline{x} + 2b + \overline{a}_1\overline{x} \geq 2b + 2\overline{a}_1\overline{x}$$

where the first inequality uses $b > \frac{1}{8}$, the second inequality uses (13) and definition of $\overline{a}_1$, and the third inequality uses $\overline{a}_1 \leq 1$. Thus

$$a_i\overline{x} < 3b \text{ for every } i = 1, ..., T+1 \tag{17}$$

Similarly note that

$$8b > 1 = \overline{x} + \left(\overline{z}_1 - x\right) + \left(1 - \overline{z}_1\right) \geq \overline{x} + 2b + 0$$

implies

$$\overline{x} < 6b \tag{18}$$

Next note that

$$4b + \overline{x}\left(a_{i-1} + a_{i+1} - 2a_i\right) \geq 4b - \overline{x}a_i > 0 \tag{19}$$

where the first inequality uses $a_{i-1} \geq a_i$ and $a_{i+1} \geq 0$, and the second uses (17).

Next note that

$$\begin{aligned}
& 4b\left(1+a_i\right) + 2 - \overline{x}\left(1+a_i\right)\left(2a_i+1\right) \\
\geq\ & 4b\left(1+a_i\right) + 2\left(2b + \overline{x}\left(1+a_i\right)\right) - \overline{x}\left(1+a_i\right)\left(2a_i+1\right) \\
=\ & \left(4b - \overline{x}\left(2a_i - 1\right)\right)\left(1+a_i\right) + 4b
\end{aligned}$$

where the inequality uses (15). If $4b \geq \overline{x}\left(2a_i - 1\right)$, then $\left(4b - \overline{x}\left(2a_i-1\right)\right)\left(1+a_i\right) + 4b > 0$.

If $4b < \overline{x}(2a_i - 1)$, then

$$\begin{aligned} & (4b - \overline{x}(2a_i - 1))(1 + a_i) + 4b \\ > \;& (4b - \overline{x}(2 - 1))\,2 + 4b \\ = \;& (6b - \overline{x})\,2 > 0 \end{aligned}$$

where the first inequality uses $a_i < 1$, and the second inequality uses (18). Thus

$$4b(1 + a_i) + 2 - \overline{x}(1 + a_i)(2a_i + 1) > 0 \text{ for every } i = 1, ..., T + 1 \tag{20}$$

Next note that

$$\begin{aligned} & 4b - 2(a_{i-1} + a_i)\overline{x} + 3(a_i + 1)(a_{i-1} + 1)\overline{x}^2 \\ = \;& 4b - 2(a_{i-1} + a_i)\overline{x} + 3(a_i + a_i a_{i-1} + a_{i-1} + 1)\overline{x}^2 \\ \geq \;& 4b - 2(a_{i-1} + a_i)\overline{x} + (4a_i + 4a_{i-1})\overline{x}^2 \\ = \;& 4b - \frac{1}{4}(a_{i-1} + a_i) + (a_{i-1} + a_i)\frac{1}{4}(4\overline{x} - 1)^2 \\ > \;& \frac{1}{2} - \frac{1}{2} + 0 = 0 \end{aligned}$$

where the inequalities use $a_i < 1$. Thus

$$4b - 2(a_{i-1} + a_i)\overline{x} + 3(a_i + 1)(a_{i-1} + 1)\overline{x}^2 > 0 \text{ for every } i = 1, ..., T + 1 \tag{21}$$

Note that $y_i = \frac{a_i}{a_i + 1} + (1 - a_i)\frac{\overline{x}}{2}$, and

$$p_i = \frac{(1 + a_i)^2\,\overline{x}(a_{i-1} - a_{i+1})(4b + \overline{x}(a_{i-1} + a_{i+1} - 2a_i))}{(2 - \overline{x}(1 + a_i)(1 + a_{i+1}))(4b(1 + a_i) + 2 - \overline{x}(1 + a_i)(2a_i + 1 - a_{i+1}))}$$

and

$$1 - p_i = \frac{(2 - \overline{x}(1 + a_{i-1})(1 + a_i))(4b(1 + a_i) + 2 - \overline{x}(1 + a_i)(2a_i + 1 - a_{i-1}))}{(2 - \overline{x}(1 + a_i)(1 + a_{i+1}))(4b(1 + a_i) + 2 - \overline{x}(1 + a_i)(2a_i + 1 - a_{i+1}))}$$

Note that $a_i \geq 0$, $\overline{x} > 0$, $a_{i-1} > a_{i+1}$, (16), (19) and (20) imply $p_i \geq 0$ for every $i = 1, ..., T+1$. Also note that (16) and (20) imply $1 - p_i \geq 0$ for every $i = 1, ..., T + 1$.

Now consider the first condition for the equilibrium

$$
\begin{aligned}
\left(\tfrac{\overline{x}+\overline{z}_1}{2}-\overline{x}-b\right)^2 = \overline{p}_1\left(\overline{y}_1-\overline{x}-b\right)^2+\left(1-\overline{p}_1\right)\overline{p}_2\left(\overline{y}_2-\overline{x}-b\right)^2+\ldots \\
+\left(1-\overline{p}_1\right)\ldots\left(1-\overline{p}_{T-2}\right)p_{T-1}\left(\overline{y}_{T-1}-\overline{x}-b\right)^2 \\
+\left(1-\overline{p}_1\right)\ldots\left(1-p_{T-1}\right)p_T\left(y_T-\overline{x}-b\right)^2 \\
+\left(1-\overline{p}_1\right)\ldots\left(1-p_{T-1}\right)\left(1-p_T\right)p_{T+1}\left(y_{T+1}-\overline{x}-b\right)^2 \\
+\left(1-\overline{p}_1\right)\ldots\left(1-p_{T-1}\right)\left(1-p_T\right)\left(1-p_{T+1}\right)\left(\tfrac{\overline{x}}{2}-\overline{x}-b\right)^2
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
&K=-p_{T-1}\left(\overline{y}_{T-1}-\overline{x}-b\right)^2-\left(1-p_{T-1}\right)p_T\left(y_T-\overline{x}-b\right)^2 \\
&-\left(1-p_{T-1}\right)\left(1-p_T\right)\left(p_{T+1}\left(y_{T+1}-\overline{x}-b\right)^2+\left(1-p_{T+1}\right)\left(\tfrac{\overline{x}}{2}-\overline{x}-b\right)^2\right)
\end{aligned} \tag{22}
$$

where K depends only on $\overline{x},\overline{z}_1,\ldots,\overline{z}_{T-1}$.

Denote the right hand side of (22) by $g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,a_{T+1}\right)$. Let

$$
h_1\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T\right)=g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,0\right)-g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},0,0\right)
$$

and

$$
h_2\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,a_{T+1}\right)=g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,a_{T+1}\right)-g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,0\right)
$$

so that

$$
K=g\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},0,0\right)+h_1\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T\right)+h_2\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,a_{T+1}\right)
$$

Since $\overline{x},\overline{a}_{T-2},\overline{a}_{T-1}$ are kept constant, we can write

$$
\widehat{K}=h_1\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T\right)+h_2\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T,a_{T+1}\right)
$$

Note that

$$
\begin{aligned}
&h_1\left(\overline{x},\overline{a}_{T-2},\overline{a}_{T-1},a_T\right) \\
=\;&\frac{\overline{a}_{T-1}a_T\overline{x}\left(\overline{a}_{T-1}-a_T\right)\left(2-\overline{x}\left(a_{T-1}+1\right)\left(\overline{a}_{T-2}+1\right)\right)}{2\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}+1\right)\right)\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T+1\right)\right)} \\
&\cdot\frac{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-\overline{a}_{T-2}+1\right)\right)}{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-a_T+1\right)\right)} \\
&\cdot\left(4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2\right)
\end{aligned}
$$

Note that $\overline{a}_{T-1}, a_T, \overline{x} > 0$, $\overline{a}_{T-1} > a_T$, (16), (20), and (21) together imply that $h_1 > 0$.

Next consider

$$\begin{aligned}
& h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1}\right) \\
= & \frac{a_T a_{T+1} \overline{x}\left(a_T - a_{T+1}\right)\left(2 - \overline{x}\left(\overline{a}_{T-1}+1\right)\left(\overline{a}_{T-2}+1\right)\right)}{2\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T+1\right)\right)\left(4b\left(1+a_{T+1}\right)+2-\overline{x}\left(1+a_{T+1}\right)\left(2a_{T+1}+1\right)\right)} \\
& \cdot \frac{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-\overline{a}_{T-2}+1\right)\right)}{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-a_T+1\right)\right)} \\
& \cdot \frac{\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T-\overline{a}_{T-1}+1\right)\right)}{\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T-a_{T+1}+1\right)\right)} \\
& \cdot \left(4b - 2\left(a_T + a_{T+1}\right)\overline{x} + 3\left(a_{T+1}+1\right)\left(a_T+1\right)\overline{x}^2\right)
\end{aligned}$$

Similarly, $a_T, \overline{x} > 0$, $a_T > a_{T+1}$, (16), (20), and (21) together imply that $h_2 > 0$ if $a_{T+1} > 0$ and $h_2 = 0$ if $a_{T+1} = 0$. Thus $\widehat{K} > 0$.

If we take $a_{T+1} = 0$ and $a_T = \overline{a}_T$, then we have KM^{T+1} equilibrium that is "equivalent" to the original KM^T equilibrium. Specifically, in this case $h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T, 0\right) = 0$ and $\widehat{K} = h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T\right)$.

Now we are going to choose $a_{T+1} > 0$ and adjust a_T. Fix $\varepsilon \in (0,1)$. Choose $\widehat{a}_T \in (0, \overline{a}_T)$ such that $h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \widehat{a}_T\right) = (1-\varepsilon)\widehat{K}$. Such $\widehat{a}_T$ exists by the intermediate value theorem since h_1 is continuous, $h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T\right) = \widehat{K}$ and $h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, 0\right) = 0$.

Next, let $a_{T+1}\left(a_T\right) = a_T - \widehat{a}_T$. Note that

$$\begin{aligned}
& h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \widehat{a}_T\right) + h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \widehat{a}_T, a_{T+1}\left(\widehat{a}_T\right)\right) \\
= & (1-\varepsilon)\widehat{K} + h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \widehat{a}_T, 0\right) = (1-\varepsilon)\widehat{K}
\end{aligned}$$

and

$$\begin{aligned}
& h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T\right) + h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T, a_{T+1}\left(\overline{a}_T\right)\right) \\
= & \widehat{K} + h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T, \overline{a}_T - \widehat{a}_T\right) > \widehat{K}
\end{aligned}$$

Since $h_1 + h_2$ is continuous, there exists $a_T \in \left(\widehat{a}_T, \overline{a}_T\right)$ such that

$$h_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T\right) + h_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1}\left(a_T\right)\right) = \widehat{K}$$

and we set $a_{T+1} = a_T - \widehat{a}_T$. ■

This completes the proof of Theorem 3. ■

Proof of Theorem 4. Suppose there is such a nontrivial equilibrium for some $b > \frac{1}{\sqrt{8}}$. By Lemmas 1 and 2, there must exist an interval of types, say $[x, z]$, such that $0 < x < z < 1$, which exit right away and get a constant action $\frac{x+z}{2}$.

Consider type x. By Lemma 2 we know that $x + b < \frac{x+z}{2}$, which implies $z - x > 2b$. Type x (or better to say x_-) on the equilibrium path must be pooled with positive probability at least for some time with types above z. Otherwise it gets actions in $[0, x]$ which are worth at most $-(x - x - b)^2 = -b^2$, while action $\frac{x+z}{2}$ is worth

$$-\left(\frac{x+z}{2} - x - b\right)^2 = -\left(\frac{z-x}{2} - b\right)^2 \geq -\left(\frac{1}{2} - b\right)^2 > -b^2$$

where the last inequality is because $b > \frac{1}{\sqrt{8}}$.

On the other hand each type $w \in [z, 1]$ that is pooled with x in equilibrium with positive probability must eventually separate from x with positive probability. Otherwise x and w will receive the same distribution over actions. The monotonicity property of the expected action in type implies that this can happen only if x and w get constant action with probability 1, which must equal $\frac{x+z}{2}$. But then these types should have exited right away by the definition of straight talk equilibrium.

Denote by $p \in (0, 1)$ the probability with which type x is completely separated from types in $[z, 1]$, and let $w \in [z, 1]$ be a type that pools with type x with the complementary probability. Note that the bliss point for w is above 1:

$$w + b \geq z + b \geq z - x + b \geq 3b > 1.$$

Thus the most type w can get after separation is $-(1 - w - b)^2$. On the other hand, type x after separation gets actions in $[0, x]$ which are worth at most $-(x - x - b)^2 = -b^2$. Denote by a and s the mean and the variance of the equilibrium lottery over actions that types x and w get conditional on pooling. Both x and w could to deviate to get action $\frac{x+z}{2}$ for sure. Thus

$$\begin{cases} -pb^2 - (1-p)\left((a - x - b)^2 + s\right) \geq -\left(\frac{x+z}{2} - x - b\right)^2, \\ -p(1 - w - b)^2 - (1-p)\left((a - w - b)^2 + s\right) \geq -\left(\frac{x+z}{2} - w - b\right)^2, \end{cases}$$

which implies

$$\begin{cases} pb^2 + (1-p)(a - x - b)^2 \leq \left(\frac{x+z}{2} - x - b\right)^2, \\ p(1 - w - b)^2 + (1-p)(a - w - b)^2 \leq \left(\frac{x+z}{2} - w - b\right)^2, \end{cases}$$

Because type x prefers action $\frac{x+z}{2}$ to action x, it must prefer action a to action $\frac{x+z}{2}$. Since $x < x+b < \frac{x+z}{2}$, it must be that $x < a < \frac{x+z}{2}$. Since $w+b > 1$, we must have $(1-w-b)^2 < \left(\frac{x+z}{2} - w - b\right)^2 < (a-w-b)^2$. Thus

$$\begin{cases} p\left(b^2 - (a-x-b)^2\right) \leq \left(\frac{x+z}{2} - x - b\right)^2 - (a-x-b)^2, \\ p\left((a-w-b)^2 - (1-w-b)^2)\right) \geq (a-w-b)^2 - \left(\frac{x+z}{2} - w - b\right)^2, \end{cases}$$

or

$$\begin{cases} p \leq \frac{\left(\frac{x+z}{2}-x-b\right)^2-(a-x-b)^2}{b^2-(a-x-b)^2} = \frac{\left(\frac{x+z}{2}-a\right)\left(\frac{x+z}{2}+a-2x-2b\right)}{(a-x)(2x+2b-a-x)}, \\ p \geq \frac{\geq(a-w-b)^2-\left(\frac{x+z}{2}-w-b\right)^2}{(a-w-b)^2-(1-w-b)^2)} = \frac{\left(\frac{x+z}{2}-a\right)\left(2w+2b-\frac{x+z}{2}-a\right)}{(1-a)(2w+2b-a-1)}. \end{cases}$$

Note that $\frac{2w+2b-\frac{x+z}{2}-a}{2w+2b-1-a} = 1 + \frac{1-\frac{x+z}{2}}{2w+2b-1-a}$ which is decreasing in w. Use it for the second inequality and let $d = \frac{z-x}{2}$:

$$\frac{(x+d-a)(2+2b-x-d-a)}{(1-a)(1+2b-a)} \leq p \leq \frac{(x+d-a)(d+a-x-2b)}{(a-x)(x+2b-a)}.$$

Subtract the lower bound from the upper bound:

$$\begin{aligned} & \frac{(x+d-a)(d+a-x-2b)}{(a-x)(x+2b-a)} - \frac{(x+d-a)(2+2b-x-d-a)}{(1-a)(1+2b-a)} \\ = \quad & \frac{(1-x)(x+d-a)(a2b+d-x+2ab-2ad+2bd-ax+dx-4b^2+x^2)}{(a-x)(1-a)(x+2b-a)(1+2b-a)}. \end{aligned}$$

All the brackets in the denominator and the ÂĚrst two brackets in the numerator are positive. Consider the last bracket in the numerator and rearrange it as follows

$$\begin{aligned} & a-2b+d-x+2ab-2ad+2bd-ax+dx-4b^2+x^2 \\ = \quad & -\left(\frac{1}{2}-4b^2\right) - 2b(d+x-a) - 2\left(\frac{1}{2}+2b-a\right)\left(\frac{1}{2}-d\right) - (1+a-2b-d-x)x. \end{aligned}$$

The first term is negative because $b > \frac{1}{\sqrt{8}}$. The second term is negative because $d + x = \frac{x+z}{2} > a$. Note that $\frac{1}{2} - d = \frac{1-z+x}{2} \geq \frac{x}{2}$, and $\frac{1}{2} + 2b - a > 0$. Thus the last two terms satisfy

$$\begin{aligned} & -2\left(\frac{1}{2}+2b-a\right)\left(\frac{1}{2}-d\right) - (1+a-2b-d-x)x \\ \leq \quad & -\left(\frac{1}{2}+2b-a\right)x - (1+a-2b-d-x)x \\ = \quad & -\left(\frac{3}{2}-d-x\right)x < 0. \end{aligned}$$

Hence the upper bound on p is below the lower bound on p, which is a contradiction. ■

Proof of Theorem 5. The welfare in KM^{T+1} equilibrium is

$$\begin{aligned}
&-p_1 (y_1 - b)^2 - (1 - p_1) p_2 (y_2 - b)^2 + ... \\
&\qquad - (1 - p_1) ... (1 - p_{T-1}) p_T (y_T - b)^2 \\
&- (1 - p_1) ... (1 - p_{T-1}) (1 - p_T) p_{T+1} (y_{T+1} - b)^2 \\
&- (1 - p_1) ... (1 - p_{T-1}) (1 - p_T) (1 - p_{T+1}) \left(\tfrac{x}{2} - b\right)^2
\end{aligned}$$

which, expressed in the equilibrium constructed in Lemma 6, is

$$\begin{aligned}
&-\overline{p}_1 (\overline{y}_1 - b)^2 - (1 - \overline{p}_1) \overline{p}_2 (\overline{y}_2 - b)^2 - ... \\
&\qquad - (1 - \overline{p}_1) ... \left(1 - \overline{p}_{T-2}\right) p_{T-1} \left(\overline{y}_{T-1} - b\right)^2 \\
&\qquad - (1 - \overline{p}_1) ... (1 - p_{T-1}) p_T (y_T - b)^2 \\
&\quad - (1 - \overline{p}_1) ... (1 - p_{T-1}) (1 - p_T) p_{T+1} (y_{T+1} - b)^2 \\
&- (1 - \overline{p}_1) ... (1 - p_{T-1}) (1 - p_T) (1 - p_{T+1}) \left(\tfrac{\overline{x}}{2} - b\right)^2
\end{aligned}$$

It can be written as

$$-\overline{U} + \overline{W} \left[\begin{array}{c} -p_{T-1} \left(\overline{y}_{T-1} - b\right)^2 - (1 - p_{T-1}) p_T (y_T - b)^2 \\ - (1 - p_T) \left(p_{T+1} (y_{T+1} - b)^2 + (1 - p_{T+1}) \left(\frac{\overline{x}}{2} - b\right)^2\right) \end{array} \right]$$

where $\overline{U}$ and $\overline{W}$ depend only on $\overline{x}, \overline{z}_1, ..., \overline{z}_{T-1}$. Note: $\overline{W} > 0$. Denote the expression inside the square brackets by $u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1})$. Let

$$v_1 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T) = u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, 0) - u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, 0, 0)$$

and

$$v_2 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1}) = u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1}) - u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, 0)$$

so that the welfare can be written as

$$-\overline{U} + \overline{W} \left[u(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, 0, 0) + v_1 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T) + v_2 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1})\right]$$

Since $\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}$ are kept constant, we can write

$$-\widehat{U} + \overline{W} \left[v_1 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T) + v_2 (\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1})\right]$$

Note that

$$
\begin{aligned}
& v_1\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T\right) \\
= & \frac{\overline{a}_{T-1} a_T \overline{x}\left(\overline{a}_{T-1}-a_T\right)\left(2-\overline{x}\left(\overline{a}_{T-1}+1\right)\left(\overline{a}_{T-2}+1\right)\right)}{2\left(4 b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2 \overline{a}_{T-1}+1\right)\right)\left(4 b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2 a_T+1\right)\right)} \\
& \cdot \frac{\left(4 b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2 \overline{a}_{T-1}-\overline{a}_{T-2}+1\right)\right)}{\left(4 b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2 \overline{a}_{T-1}-a_T+1\right)\right)} \\
& \cdot\left(4 b(1+2 \overline{x})-2\left(\overline{a}_{T-1}+a_T\right) \overline{x}-\left(3+a_T+\overline{a}_{T-1}-3 \overline{a}_{T-1} a_T\right) \overline{x}^2\right)
\end{aligned}
$$

Also note that

$$
\begin{aligned}
& v_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, a_T, a_{T+1}\right) \\
= & \frac{a_T a_{T+1} \overline{x}\left(a_T-a_{T+1}\right)\left(2-\overline{x}\left(\overline{a}_{T-1}+1\right)\left(\overline{a}_{T-2}+1\right)\right)}{2\left(4 b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2 a_T+1\right)\right)\left(4 b\left(1+a_{T+1}\right)+2-\overline{x}\left(1+a_{T+1}\right)\left(2 a_{T+1}+1\right)\right)} \\
& \cdot \frac{\left(4 b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2 \overline{a}_{T-1}-\overline{a}_{T-2}+1\right)\right)}{\left(4 b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2 \overline{a}_{T-1}-a_T+1\right)\right)} \\
& \frac{\left(4 b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2 a_T-\overline{a}_{T-1}+1\right)\right)}{\left(4 b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2 a_T-a_{T+1}+1\right)\right)} \\
& \cdot\left(4 b(1+2 \overline{x})-2\left(a_T+a_{T+1}\right) \overline{x}-\left(3+a_T+a_{T+1}-3 a_T a_{T+1}\right) \overline{x}^2\right)
\end{aligned}
$$

The welfare at the original KM^T equilibrium is the same as the welfare at KM^{T+1} such that $a_{T+1}=0$ and $a_T=\overline{a}_T$. In this case $v_2\left(\overline{x}, \overline{a}_{T-2}, \overline{a}_{T-1}, \overline{a}_T, 0\right)=0$. Thus the welfare is

$$
\begin{aligned}
& -\widehat{U}+\overline{W} v_1 \\
= & -\widehat{U}+\overline{W} \widehat{K} \frac{4 b(1+2 \overline{x})-2\left(\overline{a}_{T-1}+\overline{a}_T\right) \overline{x}-\left(3+\overline{a}_T+\overline{a}_{T-1}-3 \overline{a}_{T-1} \overline{a}_T\right) \overline{x}^2}{4 b-2\left(\overline{a}_{T-1}+\overline{a}_T\right) \overline{x}+3\left(\overline{a}_T+1\right)\left(\overline{a}_{T-1}+1\right) \overline{x}^2} \\
= & -\widehat{U}+\overline{W} \widehat{K}\left(1+2 \overline{x}-\frac{6 \overline{x}^2\left(1+\overline{x}\left(\overline{a}_T+1\right)\left(\overline{a}_{T-1}+1\right)\right)}{4 b-2\left(\overline{a}_{T-1}+\overline{a}_T\right) \overline{x}+3\left(\overline{a}_T+1\right)\left(\overline{a}_{T-1}+1\right) \overline{x}^2}\right)
\end{aligned}
$$

At the equilibrium constructed in Claim 2 (characterized by $0<a_{T+1}<a_T<\overline{a}_T$) the welfare becomes

$$
\begin{aligned}
& -\widehat{U}+\overline{W}\left[v_1+v_2\right] \\
= & -\widehat{U}+\overline{W}\left(\widehat{K}-h_2\right) \frac{4b\left(1+2\overline{x}\right)-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}-\left(3+a_T+\overline{a}_{T-1}-3\overline{a}_{T-1}a_T\right)\overline{x}^2}{4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2} \\
& +\overline{W}v_2 \\
= & -\widehat{U}+\overline{W}\widehat{K}\left(1+2\overline{x}-\frac{6\overline{x}^2\left(1+\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\right)}{4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2}\right) \\
& +\overline{W}\frac{3a_Ta_{T+1}\overline{x}^4\left(\overline{a}_{T-1}-a_{T+1}\right)\left(a_T-a_{T+1}\right)\left(2-\overline{x}\left(\overline{a}_{T-1}+1\right)\left(\overline{a}_{T-2}+1\right)\right)}{\left(4b\left(1+a_{T+1}\right)+2-\overline{x}\left(1+a_{T+1}\right)\left(2a_{T+1}+1\right)\right)} \\
& \cdot\frac{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-\overline{a}_{T-2}+1\right)\right)}{\left(4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(1+\overline{a}_{T-1}\right)\left(2\overline{a}_{T-1}-a_T+1\right)\right)} \\
& \cdot\frac{\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T-\overline{a}_{T-1}+1\right)\right)}{\left(4b\left(1+a_T\right)+2-\overline{x}\left(1+a_T\right)\left(2a_T-a_{T+1}+1\right)\right)} \\
& \cdot\frac{1}{4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2}
\end{aligned}
$$

Note that $a_T, a_{T+1}, \overline{x} > 0$, $\overline{a}_{T-1} > a_{T+1}$, $a_T > a_{T+1}$, (16), (20), and (21) together imply that the last term is strictly positive. Also note that

$$
\begin{aligned}
& \frac{d}{da_T}\left(1+2x-\frac{6\overline{x}^2\left(1+\overline{x}\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\right)}{4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2}\right) \\
= & -6\overline{x}^3\frac{4b\left(1+\overline{a}_{T-1}\right)+2-\overline{x}\left(a_{T-1}+1\right)\left(2\overline{a}_{T-1}+1\right)}{\left(4b-2\left(\overline{a}_{T-1}+a_T\right)\overline{x}+3\left(a_T+1\right)\left(\overline{a}_{T-1}+1\right)\overline{x}^2\right)^2} < 0
\end{aligned}
$$

Thus, since $\overline{W} > 0$ and $\widehat{K} > 0$, the welfare is guaranteed to increase due to $a_T < \overline{a}_T$. ■

Appendix C. Proofs for Section 4

We define

$$
f(a,x) := \frac{\left(4(a+1)b-x(a+1)^2+2\right)^2\left(4ab+2(a+1)^2x-3a+4b-1\right)}{8(a+1)^4x-4(a+1)^2(4(a+1)b+a+3)}J(x,b,1-ax) - \int_{1-ax}^{1}\frac{\left((1-s+x)(1-4b-s+x)-2x\right)\left((s-x-1)(s-x-2b)+s+x-1\right)}{2(s-x-1)^3}J(x,b,s)\mathrm{d}s,
$$

where

$$
J(x,b,s) \equiv \left(\frac{1+\frac{2b+s-x-1}{\sqrt{2}\sqrt{2b^2+x}}}{1-\frac{2b+s-x-1}{\sqrt{2}\sqrt{2b^2+x}}}\right)^{\frac{b}{\sqrt{b^2+\frac{x}{2}}}}. \tag{23}
$$

Note that (9) is equivalent to $f(a,x) = 0$. First, we note that the denominator of the first term is strictly negative for all $(x,a) \in R$, and so the function f is continuous in a, x.

Proof of Theorem 6. We have to show that no type has an incentive to deviate. The definition of M ensures that type $z = 1 - ax$ is indifferent between pooling with types $[x, z]$ and taking the gamble between revealing himself or being only known to be not in $[x, z]$, with probabilities $1 - M$ and M. The definition of (9) ensures that type x is indifferent between pooling with types $[x, z]$ and getting the lottery between $x/2$ (which occurs with probability $1 - F(1)$), and for each $s \in [z, 1]$, $\mathbf{E}[\theta \mid \theta \notin [x, s]]$ (which occurs with probability M for $s = z$, $F(\mathrm{d}s)$ otherwise). Finally, the definition of H ensures that (5) is optimal, namely, type $\theta \in [z, 1]$ wants to reveal himself at time θ if communication hasn't stopped by then. The fact that $a < 1$ ensures that $(x + z)/2 - b \in [x, z]$, and so this type $(x + z)/2 - b$ gets his bliss point by pooling with $[x, z]$; by single-crossing, it follows that all types in $[x, z]$ find it optimal to do so as well.

Finally, one must check that the single-crossing property goes in the "right" direction (which is equivalent to monotonicity of the expected action in type). This is clearly satisfied here. ■

Proof of Theorem 7. We differentiate $f(a,x)$ w.r.t. a and evaluate it at $f(a,x) = 0$. This gives

$$\left.\frac{\partial f(a,x)}{\partial a}\right|_{f(a,x)=0} = \frac{(ax + x - 1)^2(4(a+1)x - 4b - 1)\left((a+1)^2 x - 2 - 4(a+1)b\right)}{\left(4ab - 2(a+1)^2 x + a + 4b + 3\right)^2} J(x, b, 1 - ax), \quad (24)$$

where J is defined in (23). We note that, for $(x,a) \in R$,

$$(a+1)^2 x - 2 - 4(a+1)b \leq 0,$$

and so the sign of this derivative is equal to the sign of

$$1 + 4b - 4(a+1)x,$$

which is positive for $b \geq 1/4$. Because f is continuous in a, it follows that, given x, the function f admits at most one root in a (it must cross the horizontal axis "from below").

Next, we note that

$$f(0,x)=\frac{(x-2-4b)^2(1-4b-2x)}{4(4b-2x+3)}H(x,b,1),$$

and because the denominator $(4b-2x+3)$ is positive, the sign of this expression is equal to the sign of $1-4b-2x$, which is negative for $b>1/4$.

Finally, it can be verified that[14]

$$\lim_{x\to 0} xf(a,x)=\frac{8b^2\left(2(a-1)b+(a+1)(1-8b^2)\right)}{4(a+1)b+a+3},$$

and so $f(a,x)$ is positive for x small enough and a close enough to 1 if and only if $1-8b^2\geq 0$. Solving the last expression for a gives the lower bound on a, namely $a(0)=\frac{8b^2-1}{4\left(b-\frac{1}{2}\right)\left(b+\frac{1}{4}\right)}$.

Hence, for $b\geq 1/4$, $f(0,x)<0$, yet for x small enough, $f(1,x)>0$; hence a root a exists (and is unique given the above). Let $\bar{x}$ be the supremum over values of x for which such a root exists. To show that a root exists for all $x\in[0,\bar{x}]$, it suffices to invoke the implicit function theorem, and indeed, it holds that, differentiating xf w.r.t. x,

$$\left.\frac{\partial f(a,x)}{\partial x}\right|_{f(a,x)=0}<0.$$

The details of this calculation are omitted. (They rely on eliminating the logarithmic term by using the bound $\log(1+u)\geq u/(1+u)$.) This alongside the fact $f(1,0)>0$ for $b>1/\sqrt{8}$ implies that no equilibrium exists for such b. ■

Proof of Theorem 8.

[14] This is not entirely straightforward. Both terms of xf converge. The first term of xf is readily seen to converge to

$$-\frac{8b^2(2(a+1)b+1)(4(a+1)b-3a-1)}{(a+1)^2(4(a+1)b+a+3)}.$$

The second (the integral term getting subtracted) is harder to obtain. It is equal to

$$\frac{8ab^2(2(a+1)b-a)}{(a+1)^2}.$$

To get this second term, make the change of variable $s\mapsto 1-xt$, so that t varies from 0 to a, and note that the resulting integrand is a convergent series in x, with leading term

$$\frac{16b^2(bt+b-t)}{(t+1)^3}+o(x).$$

Integrating it over the range $[0,a]$ yields the result.

(Existence) Using the same notation as in the previous proof, recall that

$$\lim_{x\to 0} xf(a,x) = \frac{8b^2\left(2(a-1)b+(a+1)(1-8b^2)\right)}{4(a+1)b+a+3},$$

which is positive for all $a \in [0,1]$ if $b \leq 1/4$. Yet it is readily verified that

$$f(a,1/2) \leq -\frac{1}{10}\left(\frac{9-\sqrt{17}}{8}\right)^{\frac{1}{\sqrt{17}}} < 0,$$

which establishes that there exists (at least) one solution $x \in (0,1/2)$ to the equation $f(a,x) = 0$ in that range, for any given $a \in (0,1)$.

(Uniqueness) Follows from

$$\left.\frac{\partial f(a,x)}{\partial x}\right|_{f(a,x)=0} < 0.$$

■