CONSISTENT BELIEFS FROM THE PERSPECTIVE OF ADDITIVE REPRESENTATION

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Abstract. Recall that an assessment specifies a belief and a strategy at each information set. We note that such an assessment determines an "infinitely-less-likely relation" $\prec$ between certain pairs of nodes (as when, for example, the first node is outside, and the second node is inside, the support of the belief at an information set).

We use Farkas' Lemma to show that this relation $\prec$ has an additive representation whenever the assessment is consistent. This simple but fundamental observation allows us to simplify the proofs of two old characterizations of consistency, to repair a nontrivial fallacy in the Kreps-Wilson proofs regarding sequential equilibrium, and to develop a new characterization of consistency in terms of additive separability. (Our entire mathematical structure employs only limits, linear algebra, and Farkas' Lemma.)

In addition, we conveniently reformulate the statements of the two old characterizations. We do this by introducing "monomials."

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This paper is an extended introduction for Streufert (2007). In particular, it introduces Theorems A, B, and C, which will be stated explicitly in this paper's final section. That final section also explains where these results appear in Streufert (2007).

1. Introduction

1.1. Monomials

One of our contributions is to introduce monomials as a means of understanding the characterizations of Kreps and Wilson (1982) and Perea, Jansen, and Peters (1997). We do that in this subsection by means of an example, and both the monomials and the example will be useful when we introduce our other contributions in the next two subsections.

Imagine that you have identical twin daughters. They are ten years old, they like to giggle, and you are faced with the daunting task of putting them to bed. For some reason, you cannot distract them with songs, stories, or books, and further, you cannot reward them for good behaviour. Instead, you just notice whether or not there is noise, and then in the event of noise, you choose between two punishments, the first of which is harder on the first girl and the second of which is harder on the second girl. Note that because your twins are identical, you cannot distinguish between their giggles, and thus, you cannot distinguish between the first girl giggling, the second girl giggling, and the two of them giggling simultaneously. All you can do is to notice whether or not there is noise.

That was a verbal description of the game (without payoffs) that appears in Figure 1(a). The three players are Twin 1, Twin 2, and Parent. The first twin chooses between G (giggle) and S (sleep); the second twin chooses between g (giggle) and s (sleep); and the parent chooses between δ (the punishment that is harder on the first twin) and ε (the punishment that is harder on the second twin). Note that the nodes are identified with the sequence of actions taken to reach them

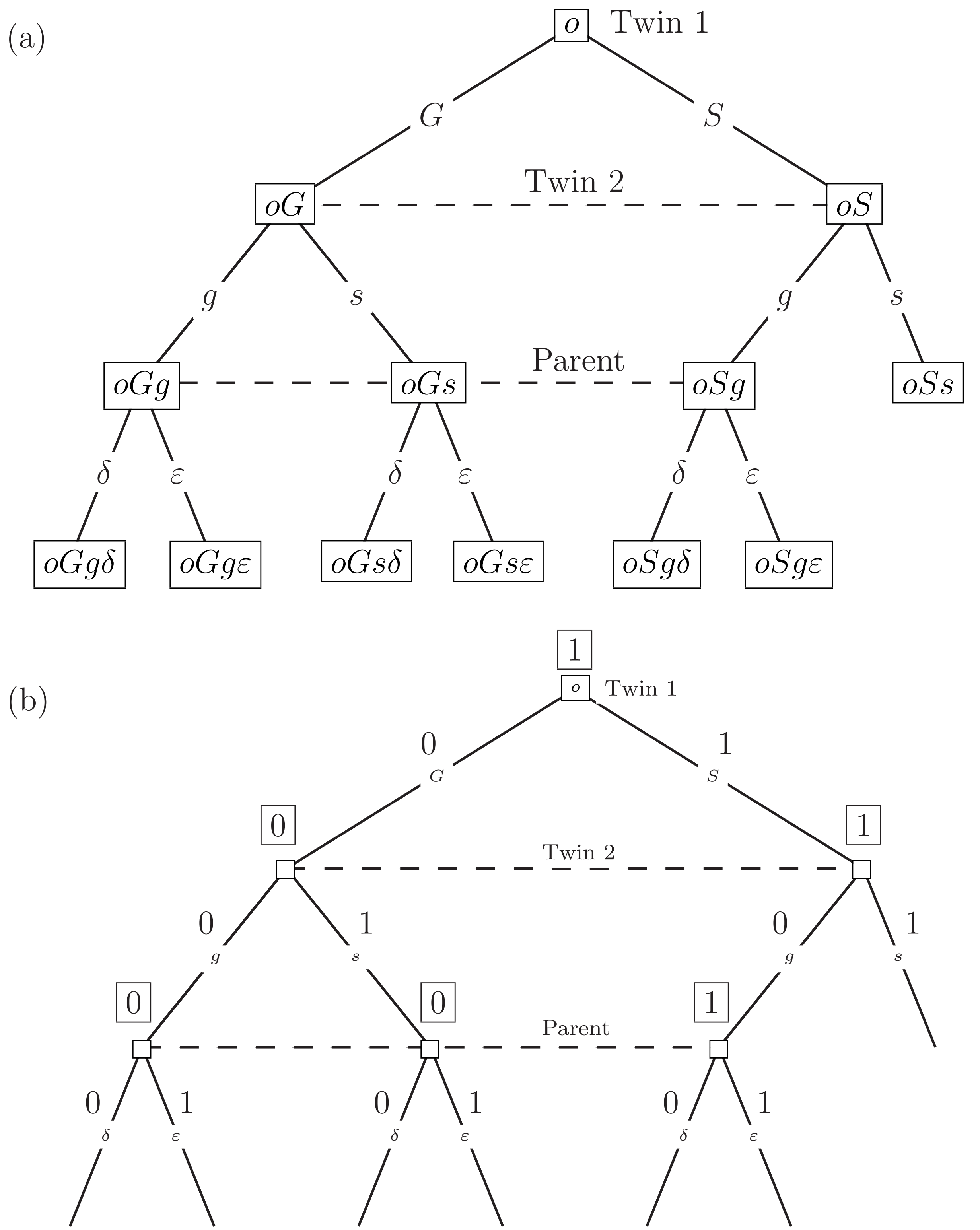

Figure 1. (a) The game (without payoffs) in full detail. (b) An assessment in which both girls sleep and the parent would blame the second girl for any noise (the probabilities on δ and ε are irrelevant for consistency).

(o is the original node). Accordingly, future figures, like Figure 1(b), will not explicitly label any nodes. (This game resembles Kreps and Ramey (1987, Figure 1).)

In the story, the girls might go to sleep in spite of the fact that you could never tell who was giggling. For example, it might be that both girls know (a) that you would believe that the second girl alone was at fault, (b) that this belief would induce you to choose the second punishment, and (c) that the threat of the second punishment is sufficient to make them both behave. Or, alternatively, it might be that both girls know (a) that you would believe that both girls are giggling simultaneously, (b) that this would induce you to randomize in some way between the two punishments, and (c) that the threat of this randomized punishment is sufficient to make them both behave. (In each example, steps (b) and (c) depend on exogenous payoffs which have been left unspecified.)

Although either scenario makes sense superficially, the logic is murky because it is unclear how the parent's belief may or may not accord with the girls' behaviour. Since the girls' good behaviour implies zero probability at each of the nodes oGg, oGs, and oSg, we might say that any belief over these three nodes accords with the girls' behaviour. Such thinking would admit both of the above scenarios. Ordinary probability theory can provide no further guidance.

Alternatively, we might be more restrictive. We might think that both girls giggling is infinitely less likely than either of the two giggling alone, simply because the coincidence of two zero-probability events seems infinitely less likely than either zero-probability event alone. Such thinking would rule out the second of the two scenarios.

This restriction is implied by the concept of consistency which Kreps and Wilson (1982, henceforth `KW`) introduced as part of their path-breaking sequential-equilibrium concept. Recall that a sequential equilibrium is an assessment (that is, a list of strategies and beliefs) that is both consistent and sequentially rational. Consistency means that the assessment is the limit of a sequence of full-support (i.e. positive-valued) assessments, each of which has beliefs that are derived from its strategies by means of the conditional-probability law of ordinary probability theory (the law always works for full-support assessments).

For example, consider the assessment that is specified by the numbers in Figure 1(b): the unboxed numbers are the strategies and the boxed numbers are the beliefs. For instance, the boxed "1" at node oSg states that the parent would believe that the second girl alone is at fault, as in our first scenario. This assessment is consistent because it is the limit of the sequence of assessments that are specifed in Figure 2(a), and because, at any n, the beliefs in Figure 2(a) are derived from the strategies by the ordinary conditional-probability law.

Figure 2(a) typifies the manner in which economists have learned to verify consistency. First, one assigns to each action a a monomial of the form $c(a)n^{e(a)}$, where the coefficient $c(a)$ is a positive real, the exponent $e(a)$ is a nonpositive integer, and the n will become the sequence index (e.g., the numerators in Figure 2(a) assign the monomial n^{-2} to action G and the monomial $1 = n^0$ to action S). Second, for a fixed n, the monomials are normalized to create strategies (e.g., the strategy of the first girl is $\pi_n(G) = \frac{n^{-2}}{1+n^{-2}}$ together with $\pi_n(S) = \frac{1}{1+n^{-2}}$). Third, again for a fixed n, the beliefs are derived from the strategies by the ordinary conditional-probability law. For example, the probability $\mu_n(oSg)$ that the parent places on only the second girl giggling is derived as

$$\frac{\pi_n(S)\pi_n(g)}{\pi_n(G)\pi_n(g) + \pi_n(G)\pi_n(s) + \pi_n(S)\pi_n(g)} = \frac{n^{-1}}{n^{-1} + n^{-2} + n^{-3}} \ .$$

And finally, it is verified that the assessment in question is the limit of this sequence of assessments (e.g., Figure 1(b) is the limit of Figure 2(a)'s assessments.)

The third of these steps can be tedious because one must simplify a fraction whose denominator is the sum of several products of fractions. This denominator is particularly ugly when a game's players do

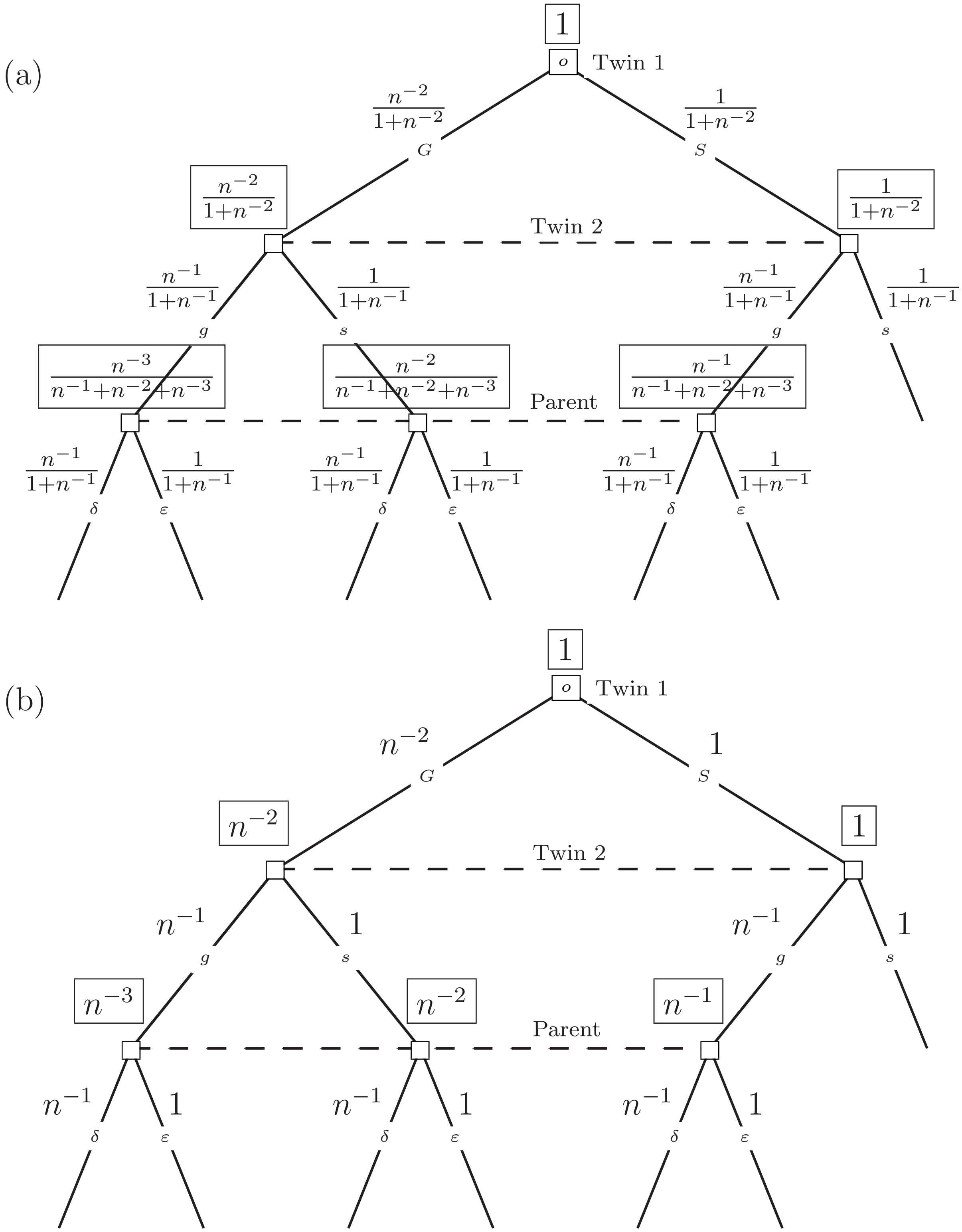

FIGURE 2. Two ways to show the consistency of Figure 1(b)'s assessment: (a) uses the definition of consistency (or alternatively the easy half of Theorem A), while (b) uses the easier half of Theorem B.

not move in a predetermined order, for then, the denominator's many products often have different numbers of factors and hence different denominators. And yet further, the ensuing algebra is not only tedious but also frustrating because it often finishes with second-order nuisance terms in the numerator.

Theorem B shows that all this tedium and complexity can be avoided. The trick is to consider a monomial $c(a)n^{e(a)}$ as an extended sort of probability number. Essentially, monomials with zero exponents express positive probabilities; and monomials with negative exponents express different levels of zero probability (lesser, i.e., more negative, exponents express lesser zero probabilities).

As in the standard technique, begin by assigning monomials to actions, but this time, be sure to impose two constraints on that assignment: (1) if the assessment plays an action a with probability $\pi(a) > 0$, set the monomial $c(a)n^{e(a)}$ equal to $\pi(a)$ (the exponent is zero), and (2) if $\pi(a) = 0$, choose a monomial so that its exponent $e(a)$ is negative. Then, as in the boxed monomials of Figure 2(b), derive a monomial for each node by taking the product of the monomials of the actions leading to it. These derived (boxed) monomials determine beliefs by an extended sort of conditional-probability law: the support of the belief over each information set is determined by the monomials with the highest exponent, and then the probability is distributed across that support in proportion to the monomials' coefficients. Theorem B shows that any assessment which can be derived from monomials in this fashion is consistent.

Further, and much more subtly, Theorem B also shows that the existence of such monomials is *necessary* for consistency. This is a very useful result.

For instance, recall our second scenario in which the parent would believe that both girls are giggling simultaneously. If one tried to verify that this assessment was consistent, one would repeatedly assign monomials and keep coming up empty. This would leave one with the suspicion that the assessment was not consistent, but without a means of proving this suspicion. In particular, the definition of consistency would be of little use since it would require ruling out all elements in the infinite-dimensional space of assessment sequences.

The *necessity* of monomials makes this proof of inconsistency a simple matter. No matter how you assign monomials to G and S, it must be that $e(G)$ is less than $e(S)$ because the first girl goes to sleep. Thus the exponent $e(G)+e(g)$ on the monomial for oGg must be less than the exponent $e(S)+e(g)$ on the monomial for oSg. Hence oGg cannot have the highest exponent in the parent's information set $\{oGg, oGs, oSg\}$, and hence any consistent belief must place zero probability on both girls giggling simultaneously.

As stated at the outset, Theorem B and its cousin Theorem A are reformulations of results in Kreps and Wilson (1982) and Perea, Jansen, and Peters (1997). This reformulation is the first of our contributions, and is certainly the least impressive from a technical standpoint.

Nonetheless, the reformulation contributes modestly to the agenda of finding the "smallest" system of nonstandard numbers which is sufficiently rich to express concepts such as sequential equilibrium and perfect equilibrium (an overview appears in Hammond (1994)). Although we have nothing to say about other equilibrium concepts, our reformulation does show that the monomials are sufficiently rich for sequential equilibrium. This fact might have been overlooked simply because the set of monomials is not an extension of the real number system. Rather, it is a (subset of) a group with only one operation (we do not use addition).

1.2. The Perspective of Additive Representation

Our second and most important contribution is to introduce the idea that consistency can be viewed from the perspective of additive representation. Unfortunately, this new perspective will take a moment to explain. It is an unfamiliar combination of elementary observations.

We proceed gradually in four steps, the first two of which are independent from one another.

(a) Exponents as an Additive Representation. We first observe that the exponents of monomials can be used to represent an ordering over nodes. In particular, take any assignment e of exponents to actions and construct the relation $\preceq^e$ by

$$(\forall t,r)\ t \preceq^e r \text{ iff } \Sigma_t e \leq \Sigma_r e, \tag{1}$$

where $\Sigma_t e$ denotes the sum of exponents assigned to the actions leading to t. For example, the exponents e of Figure 2(b)'s monomials result in $\Sigma_{oG}e = e(G) = -2$, $\Sigma_{oS}e = e(S) = 0$, and $\Sigma_o e = \Sigma\varnothing = 0$, and thus $oG \prec^e o$ and $oS \approx^e o$. Further, that same e results in

$$\Sigma_{oGg}e = e(G){+}e(g) = -2{+}{-1} = -3$$
$$\Sigma_{oGs}e = e(G){+}e(g) = -2{+}0 = -2$$
$$\Sigma_{oSg}e = e(S){+}e(g) = 0{+}{-1} = -1\ ,$$

and thus $oGg \prec^e oGs$, $oGg \prec^e oSg$, and $oGs \prec^e oSg$.

We regard such a representation as being "additive" across information sets (i.e., agents). To make this "additivity" more familiar, this paragraph momentarily digresses to consider a closely related representation over a standard Cartesian product. First, note that each of the 13 nodes in Figure 1(a) can be identified with one of the 27 tuples in

$$\{G,\ S,\ \text{no-Twin-1-action}\}\times$$
$$\{g,\ s,\ \text{no-Twin-2-action}\}\times$$
$$\{\delta,\ \varepsilon,\ \text{no-Parent-action}\}\ .$$

Second, extend the assignment e into e^* by setting e^* equal to zero at the three new "null" actions. And finally, define $\preceq^{e*}$ by applying (1), with e^* rather than e, over the 27 tuples rather than the 13 nodes. This ordering $\preceq^{e*}$ is represented additively across information sets in a manner that is compatible with standard preference theory. If one identifies the 13 nodes with the corresponding 13 tuples, our ordering

$\preceq^e$ is merely a restriction of this additively separable $\preceq^{e*}$. (The tuples, the Cartesian product, and the extensions e^* and $\preceq^{e*}$ do not appear outside this paragraph.)

Now momentarily forget about exponents e and the ordering $\preceq^e$. (This is an abrupt but necessary break.)

(b) An Assessment's Infinitely-Less-Likely Relation $\preceq$. Start afresh with an assessment and notice that an assessment determines the relative probability between many (but not all) pairs of nodes. Some (but not all) of this information is embodied in the relation $\preceq$ that we are about to define.

An assessment's strategies determine the probability of any node relative to its predecessor. That relative probability is just the ordinary probability that the strategy assigns to the action that leads to the node from its predecessor. If that probability is zero, the node is infinitely-less-likely than its predecessor and the pair belongs to $\prec$. On the other hand, if that probability is positive, the pair belongs to $\approx$. For example, Figure 1(b)'s assessment induces $oG \prec o$ and $oS \approx o$ (those two facts are represented by the $\prec$ and $\approx$ (actually $=$) at the top of Figure 3).

An assessment's beliefs determine the probability of any node relative to any other node that is both in the same information set and in the belief's support. If the first node is outside the belief's support and the second node is both in the same information set and in the belief's support, then the first node is infinitely-less-likely than the second and the pair belongs to $\prec$. On the other hand, if two distinct nodes are both in the same information set and in the belief's support, then the pair belongs to $\approx$. For example, Figure 1(b)'s assessment induces $oGg \prec oSg$ and $oGs \prec oSg$ (those two facts are represented by the $\prec$'s near the lower dotted curves in Figure 3).

An assessment's weak infinitely-less-likely relation $\preceq$ is essentially the union of this $\prec$ and this $\approx$ (a formal definition of $\preceq$ appears in Streufert (2007) Section 3.1). Note that the $\preceq$ of a typical assessment

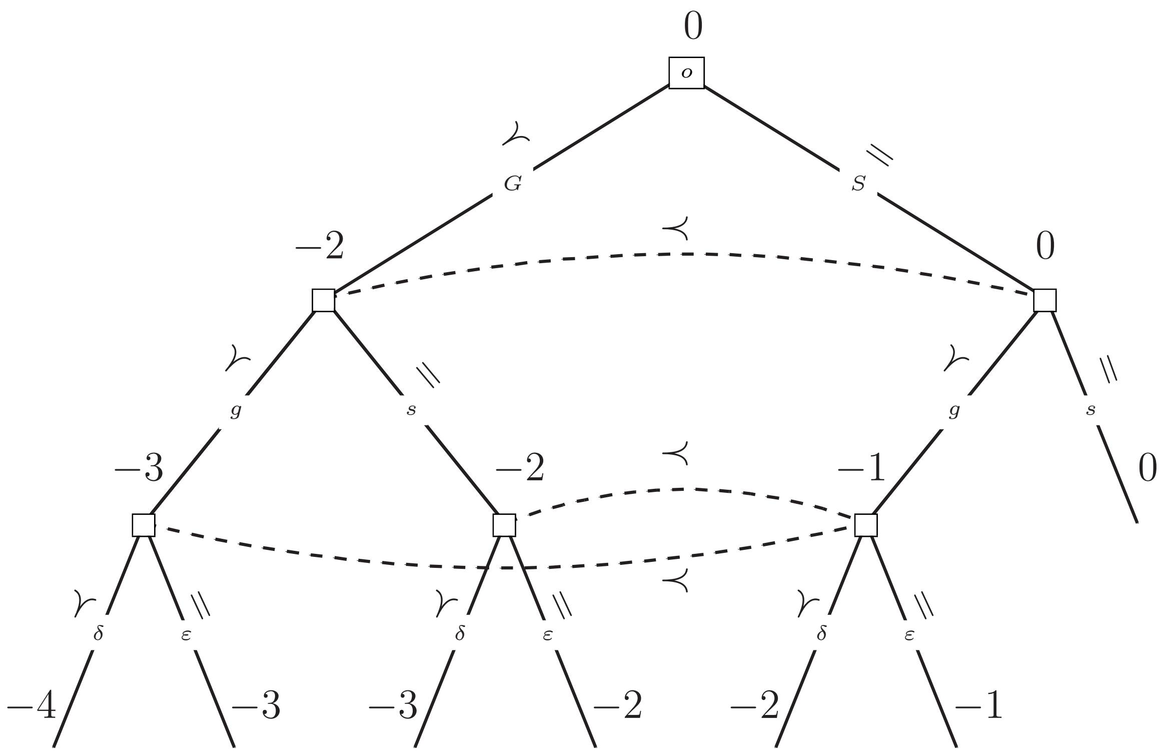

FIGURE 3. The infinitely-less-likely relation $\preceq$ of Figure 1(b)'s assessment, being represented by the sum $\Sigma_{(\cdot)}e$ of exponents from Figure 2(b)'s monomials. (The $\approx$'s appear as $=$'s due to a software problem.)

is incomplete. For example, Figure 3 shows the $\preceq$ induced by Figure 1(b)'s assessment, and this $\preceq$ makes no comparison between oGg and oGs. (Further, there is no attempt to make an assessment's $\preceq$ transitive or reflexive. Rather, $\preceq$ is made as small as possible.)

(c) Consistency. We now come to the crux of the matter: If an assessment is consistent, there is an e whose $\preceq^e$ is an extension of the assessment's infinitely-less-likely relation $\preceq$. In other words, every consistent assessment admits an additive representation of (an extension of) its infinitely-less-likely relation.

For example, consider Figure 1(b)'s assessment. Because this assessment is consistent, there should be an e such that $\preceq^e$ is an extension of the assessment's $\preceq$ in the standard sense that

$$\text{(2a)} \qquad (\forall t, r)\ t \prec r \text{ implies } t \prec^e r \text{ and}$$

$$\text{(2b)} \qquad (\forall t, r)\ t \approx r \text{ implies } t \approx^e r\ .$$

In fact, such an e is provided by Figure 2(b)'s exponents. For instance, we have seen that

$$oG \prec o,\ oS \approx o,$$
$$oGg \prec oSg, \text{ and } oGs \prec oSg\ ,$$

and also that

$$oG \prec^e o,\ oS \approx^e o,$$
$$oGg \prec^e oGs,\ oGg \prec^e oSg, \text{ and } oGs \prec^e oSg$$

(it happens that $\preceq^e$, but not $\preceq$, can compare oGg and oGs).

Measure theory often uses an alternative definition of additive representation which would say that e additively represents $\preceq$ if

$$\text{(3a)} \qquad (\forall t, u)\ t \prec u \text{ implies } \Sigma_t e < \Sigma_u e \text{ and}$$

$$\text{(3b)} \qquad (\forall t, u)\ t \approx u \text{ implies } \Sigma_t e = \Sigma_u e\ .$$

Although (1) and (3) are equivalent whenever the binary relation is complete, the distinction is important here because the infinitely-less-likely relation of an assessment is typically incomplete.

Conveniently, by the definition (1) of $\preceq^e$, statements (2) and (3) are equivalent. Thus, e represents (1) $\preceq^e$ which extends (2) $\preceq$ iff e represents (3) $\preceq$. Hence our central observation can be reformulated without reference to $\preceq^e$: if an assessment is consistent, then its infinitely-less-likely relation $\preceq$ has an additive representation in the sense of (3).

(d) Intuition. Although this subsection has roamed across preference theory, game theory, and measurement theory, we have actually arrived at something that is very close to the way that many of us first learned to understand consistency.

By working examples like the one in Subsection 1.1, many of us learned that the trick to verifying the consistency of a given assessment was to cleverly assign negative exponents to zero-probability actions in such a way that they generate the support of the assessment's beliefs at zero-probability information sets.

In particular, if one node t is outside the belief's support and another node r is both in the same information set and in the belief's support, then the exponents must have been cleverly chosen so that the sum of the exponents on the path to t is less than the sum of the exponents on the path to r. This is part of what (3a) requires: the hypothesis of the previous sentence implies $t \prec r$ and the conclusion of the previous sentence is $\Sigma_t e < \Sigma_r e$.

Further, if two nodes t and r are both in the same information set and in the belief's support, then the exponents must have been cleverly chosen so that the sum of the exponents on the path to r is equal to the sum of the exponents on the path to r. This is part of what (3b) requires: the hypothesis of the previous sentence implies that $t \approx r$ and the conclusion of the previous sentence is $\Sigma_t e = \Sigma_r e$.

As many of us learned, finding such exponents comes close to verifying the consistency of the assessment (via the definition of consistency or via the easy half of Theorems A or B). Our central observation is the converse: every consistent assessment admits such exponents (this is the crux of the difficult half of Theorems A and B).

1.3. Contributions from the New Perspective

Again, our central point is that if an assessment is consistent, then its infinitely-less-likely relation $\preceq$ can be represented additively across information sets. At one level, this new observation is a corollary of Theorem B, which is in turn a reformulation of results in `KW` and `PJP`.

Yet a deeper level, we would suggest that this observation is fundamental. This new perspective suggests four tasks: (a) to derive an additive representation in the simplest possible way, (b) to compare our derivation with the `KW` proofs, (c) to compare our derivation with the `PJP` proofs, and (d) to wonder if additive representation can clarify the concept of consistency itself.

(a) Our Derivation of Additive Representation. It is surprizingly easy to show that a consistent assessment's $\preceq$ has an additive representation.

While economists are more familiar with binary relations having continuous domains, there is a large literature in mathematical psychology and measurement theory which studies binary relations having discrete domains. A classic result there is Scott (1964)'s Theorem, which states that a relation has an additive representation iff it satisfies a collection of cancellation laws. Our proof intuitively derives (a slight variation of) these cancellation laws from the definition of consistency while using nothing more complicated than limits (Streufert (2007), first full paragraph of page 18).

Scott's classic result is easily proven as a corollary of Farkas' Lemma, and accordingly, our proofs use nothing more complicated than limits, Farkas' Lemma, and linear algebra (none of our variables, other than the sequence index n, equals or tends to infinity).

(b) The KW Proofs. The new perspective of additive representation reveals a nontrivial fallacy in the `KW` proofs.

To be somewhat more specific, Theorem B extensively reformulates the combination of `KW` Lemmas A1 and A2. The proof of `KW` Lemma A1 defines a relation $\dot{>}$ that can be regarded as an extension of our $\preceq$. Then, from our new perspective (and in spite of their very different terminology), we can now see that they assert, but do not prove, that their relation has an additive representation. This assertion is critical to their proof of Lemma A1. Streufert (2007)'s Sections 4.1 and 4.2 explain the fallacy in more detail, and then its Section 4.3 derives `KW` Lemmas A1 and A2 from our Theorem B.

This repair is important because `KW` Lemmas A1 and A2 are critical to the two `KW` theorems which derive the generic finiteness of the set of sequential-equilibrium outcomes and the generic equality of that set to the set of perfect-equilibrium outcomes. These well-known theorems are textbook material (e.g., Fudenberg and Tirole (1991), Theorems 8.1 and 8.5).

(c) The PJP Proofs. Our elementary derivation of additive representation allows us to provide a simpler alternative to the PJP proofs.

To be more specific, Theorem B is a reformulation of a slight sharpening of `PJP` Theorem 3.1, and its cousin Theorem A is a reformulation of a slight sharpening of `PJP` Corollary 3.3. Our proofs of these results are (subjectively) more intuitive in the sense that they naturally construct an infinitely-less-likely relation and then follow Scott's classical representation theorem to derive its additive representation.

Further, our proofs are more economical in their use of mathematics. In particular, we rely on Farkas' Lemma while PJP relies upon the Separating Hyperplane Theorem (PJP page 244). In hindsight, this simplification rings true because the Separating Hyperplane Theorem can be usefully regarded as an analytic generalization of Farkas' Lemma (Ziegler (1995), page 40).

(d) A New Characterization of Consistency. Finally, the new perspective of additive representation does provide an insight into the nature of consistency itself.

Roughly speaking, Theorem C states that an assessment is consistent iff [1] its infinitely-less-likely relation $\preceq$ can be represented additively across information sets (the necessity of this is our central observation), and [2] certain finite relative probabilities can be specified multiplicatively across information sets. Part [1] is like the conventional notion of independence from preference theory, and part [2] is like the conventional notion of independence from ordinary probability theory. Thus the two parts together seem to be a new sort of stochastic independence for relative probabilities, that has been defined in the specific context of an extensive-form game. Or in other words, the two parts together provide an extended concept of independence which says what it means for the agents' (i.e., information sets') strategies to be "independent" both on and off the equilibrium path. In this sense, Theorem C is able to characterize consistency as a new sort of stochastic independence.

This leads in several directions.

First, recall the twins. In light of Theorem C, consistency requires that the impossible event of Twin 1 giggling is "independent" of the impossible event of Twin 2 giggling. And in accord with our earlier discussion, that "independence" implies that the joint event of both girls giggling simultaneously is infinitely less likely than either of the two joint events in which one girl giggles and the other does not. In brief, consistency rules out "correlation" between agents: it simplifies things.

Second, Theorem C hints at how this entire paper is a natural outgrowth of Kohlberg and Reny (1997), who focussed on relative probabilities and stochastic independence. In particular, their appealing focus on relative probability eventually led us to the infinitely-less-likely relation $\preceq$ that is central to this paper.

And finally, Theorem C makes one to wonder how to formulate stochastic independence for relative probabilities, in a general setting apart from extensive-form games. Some initial work appears in Streufert (2006a), whose approach can again be traced to Kohlberg and Reny (1997).

2. Theorems

Theorem A appears within Streufert (2007, Theorem 2.1) as the equivalence of consistency with that theorem's condition (b). As in `KW`, ρ denotes the game's exogenous chance probabilities (if any), a strategy $\pi{:}A{\to}[0,1]$ is defined over the set A of actions a, and a belief $\mu{:}X{\to}[0,1]$ is defined over the set X of decision nodes x. Further, $\Pi_x\rho{\cup}\pi$ denotes the probability of reaching node x, that is, it is the product of the chance probabilities ρ and the strategies π on the path to node x (the "$\cup$" notation may be of questionable value).

Theorem A. *An assessment (π, μ) is consistent if and only if there exist $c{:}A{\to}(0,1]$ and $e{:}A{\to}\{...\text{-}2,\text{-}1,0\}$ such that (π, μ) is the limit of the sequence $\{(\pi_n, \mu_n)\}_n$ defined by*

$$(\forall a)\ \pi_n(a) = \frac{c(a)n^{e(a)}}{\Sigma_{a'\in A\circ A^{-1}(a)}\, c(a')n^{e(a')}}\ \textit{and}$$

$$(\forall x)\ \mu_n(x) = \frac{\Pi_x\rho{\cup}\pi_n}{\Sigma_{x'\in H(x)}\,\Pi_x\rho{\cup}\pi_n}\ .$$

Theorem B appears within Streufert (2007, Theorem 2.1) as the equivalence of consistency with that theorem's condition (a). Here $\Sigma_x e$ is the sum of the exponents e on the path to node x, and $\Pi_x\rho{\cup}c$ is the product of chance probabilities ρ and coefficients c along the same path (the multiplication of monomials is a convenient way of calculating these terms).

Theorem B. *An assessment (π, μ) is consistent if and only if there exist $c{:}A{\to}(0,1]$ and $e{:}A{\to}\{...\text{-}2,\text{-}1,0\}$ such that*

$$(\forall a)\ \pi(a) = \begin{pmatrix} c(a) & \textit{if } e(a)=0 \\ 0 & \textit{if } e(a)<0 \end{pmatrix}\ \textit{and}$$

$$(\forall x)\ \mu(x) = \begin{pmatrix} \dfrac{\Pi_x\rho{\cup}c}{\Sigma_{x'\in H^e(x)}\,\Pi_{x'}\rho{\cup}c} & \textit{if } x\in H^e(x) \\ 0 & \textit{if } x\notin H^e(x) \end{pmatrix}$$

where $H^e(x) = \textit{argmax}\{\,\Sigma_{x'}e \,|\, x'{\in}H(x)\,\}$.

Both the definition of $\preceq$ and Theorem C are buried within the proof of Streufert (2007, Theorem 2.1). In particular, the definition of $\preceq$ and the derivation of an additive representation for $\preceq$ appear in that paper's Subsection 3.1. Here that derivation of an additive representation appears prominently, in the following theorem, as the necessity of Condition [1]. That necessity is our paper's key observation.

THEOREM C. *An assessment (π, μ) is consistent if and only if [1] its weak infinitely-less-likely relation $\preceq$ has an additive representation (in the sense of (3)) and [2] there exists $c{:}A{\rightarrow}(0,1]$ such that $c(a) = \pi(a)$ whenever $\pi(a)$ is positive and $\Pi_x\rho{\cup}c/\Pi_y\rho{\cup}c = \mu(x)/\mu(y)$ whenever x and y are decision nodes such that $x \approx y$.*

References

FUDENBERG, D., AND J. TIROLE (1991): *Game Theory.* MIT Press.

HAMMOND, P. J. (1994): "Elementary Non-Archimedean Representations of Probability for Decision Theory and Games," in *Patrick Suppes: Scientific Philosopher, Volume 1*, ed. by P. Humphreys, pp. 25–61. Kluwer.

KOHLBERG, E., AND P. J. RENY (1997): "Independence on Relative Probability Spaces and Consistent Assessments in Game Trees," *Journal of Economic Theory*, 75, 280–313.

KREPS, D. M., AND G. RAMEY (1987): "Structural Consistency, Consistency, and Sequential Rationality," *Econometrica*, 55, 1331–1348.

KREPS, D. M., AND R. WILSON (1982): "Sequential Equilibria," *Econometrica*, 50, 863–894.

PEREA, A., M. JANSEN, AND H. PETERS (1997): "Characterization of Consistent Assessments in Extensive Form Games," *Games and Economic Behavior*, 21, 238–252.

SCOTT, D. (1964): "Measurement Structures and Linear Inequalities," *Journal of Mathematical Psychology*, 1, 233–247.

STREUFERT, P. A. (2006a): "Products of Several Relative Probabilities," University of Western Ontario, Economics Department Research Report 2006-01, 31 pages.

——— (2006b): "Characterizing Consistency by Monomials and by Product Dispersions," University of Western Ontario, Economics Department Research Report 2006-02, 27 pages.

——— (2006c): "A Comment on 'Sequential Equilibria'," University of Western Ontario, Economics Department Research Report 2006-03, 10 pages.

——— (2007): "Characterizing Consistency by Monomials," 34 pages.

ZIEGLER, G. M. (1995): *Lectures on Polytopes.* Springer-Verlag.