ACTION-BASED GAME FORMS

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Abstract. A "game form" is a subset of the objects in a game: it includes nodes, branches, actions, and information sets, but excludes players, chance probabilities, and payoffs. To define such a game form, it is standard to begin with nodes and branches, and then to label branches with actions. Instead we begin with actions, and then define each node as a set of actions. We show that such action-based game forms are equivalent to node-based game forms having perfect agent recall. The benefit is that, in our opinion, action-based game forms are more elegant.

1. Introduction

Consider the standard definition of a game, as in Mas-Colell, Whinston, and Green (1995) for example. This standard definition begins with a "game tree" consisting of nodes (i.e. vertices) and edges (i.e. branches). Next it labels the tree's edges with actions, and groups the tree's nonterminal nodes into agents (i.e. information sets). Finally, it groups the agents into players, assigns probabilities to the chance player's actions, and assigns payoffs to strategic players at each terminal node.

If one were to interrupt this standard definition after its first two steps, one would be left with nodes, edges, actions, and agents. We refer to such a collection of objects as a "game form," and note that our usage of this term is broadly consistent with its appearance elsewhere, as in Gibbard (1978) for example.

We will refer to the standard game form as a "node-based" game form because its definition begins with the game tree's nodes. In contrast, this paper proposes an "action-based" game form which starts with actions and then defines nodes as sets of actions.

Date: June 29, 2011.

The author would like to thank Greg Pavlov, Al Slivinski, and the other members of the University of Western Ontario Theory Workshop for their helpful comments.

Our paper's theorems demonstrate that the two specifications are equivalent. In particular, we show that every action-based game form determines a node-based game form having perfect agent recall, and conversely, that every node-based game form having perfect agent recall determines an action-based game form. This "perfect agent recall" means that no path leaves an agent (i.e. information set) and later returns to the same agent. It is a substantial restriction which is nonetheless weaker than the perfect recall assumed by much of the literature, including Kreps and Wilson (1982), for example.

Since beauty must always remain in the eye of the beholder, we can never hope to prove that action-based game forms are more elegant than node-based game forms. However, we can mention two aspects of action-based game forms that we find particularly appealing.

First, many economists are well acquainted with the idea of a topology, which is a collection of sets, each of which contains elements from some underlying domain. Similarly, many are acquainted with the idea of a σ-algebra, which is again a collection of sets, each drawing its elements from some underlying domain. Here, a collection of nodes is like a topology or σ-algebra: it is a collection of sets, each drawing its elements from some underlying domain, which in this case, is the set of actions.

Second, game theory is filled with the idea of multiplying together the probabilities of the actions that lead to a node. This operation appears whenever an agent calculates beliefs, and again whenever an agent calculates the expected payoffs of available options. And yet, these routine products-over-actions are difficult to express formally within the standard node-based formulation because it is difficult to express formally the set of actions that lead to a node. Within our action-based formulation, the set of actions leading to a node is identical with the node itself, and hence, these routine products-over-actions can be transparently expressed.

2. Defining an action-based game form

Let A be a nonempty finite set of *actions* a. Then let T be a collection of *nodes* t, each of which satisfies $t \subseteq A$. Or, in other words, let $T \subseteq \mathcal{P}(A)$, where $\mathcal{P}(A)$ denotes the set of all subsets of A.

We now take an unexpected step. We derive from A and T the feasibility correspondence F that is already implicit these two sets (later

All nonterminal nodes (repeated as needed) $t{\in}T{\sim}Z$	All actions $a{\in}A$	All noninitial nodes (each listed once) $t{\cup}\{a\}{\in}T{\sim}\{\{\}\}$
$\{\}$	r_1	$\{r_1\}$
$\{\}$	d_1	$\{d_1\}$
$\{r_1\}$	r_2	$\{r_1,r_2\}$
$\{r_1\}$	d_2	$\{r_1,d_2\}$
$\{r_1,r_2\}$	d_3	$\{r_1,r_2,d_1\}$
$\{r_1,r_2\}$	r_3	$\{r_1,r_2,r_2\}$

TABLE 2.1. Tabular portrayal of the action tree (A,T) defined by $A = \{d_1, r_1,\ d_2, r_2,\ d_3, r_3\}$ and $T = \{\ \{\}, \{d_1\}, \{r_1\}, \{r_1,d_2\}, \{r_1,r_2\}, \{r_1,r_2,d_3\}, \{r_1,r_2,r_3\}\ \}$.

we will impose an assumption of F). In particular, let F be the correspondence from T into A that satisfies

$$(\forall t)\ F(t) = \{\ a \mid a{\notin}t \text{ and } t{\cup}\{a\}{\in}T\ \}\ .$$

Since every action a in $F(t)$ can be combined with the node t to produce a new node $t{\cup}\{a\}$, the set $F(t)$ can be understood as the set of actions that are *feasible* from t. Note that

$$F = \{\ (t,a) \mid a{\notin}t \text{ and } t{\cup}\{a\}{\in}T\ \}$$

since the notations $a{\in}F(t)$ and $(t,a){\in}F$ are equivalent.

This notation departs from the literature in the sense that it is standard to let the symbol A denote not only the set of actions (as above) but also the feasibility correspondence (which is denoted here by F). We need to make this distinction explicit, and accordingly, $F(t)$ rather than $A(t)$ will denote the set of actions that are feasible from t.

Please see the examples defined in the captions of Tables 2.1 and 2.2. The rows of the table are in one-to-one correspondence with the elements (t,a) of F: the first column of the row contains t and the second column of the row contains a. For example, in Table 2.1, $F = \{\ (\{\},d_1), (\{\},r_1), (\{r_1\},d_2), (\{r_1\},r_2), (\{r_1,r_2\},d_3), (\{r_1,r_2\},r_3)\ \}$.

As a matter of convention, we will continue to denote the empty set by $\{\}$ rather than $\varnothing$ when we are regarding the empty set as a node. This convention is already being used in the two examples.

All nonterminal nodes (repeated as needed) $t{\in}T{\sim}Z$	All actions $a{\in}A$	All noninitial nodes (each listed once) $t{\cup}\{a\}{\in}T{\sim}\{\{\}\}$
$\{\}$	g_1	$\{g_1\}$
$\{\}$	s_1	$\{s_1\}$
$\{g_1\}$	g_2	$\{g_1,g_2\}$
$\{g_1\}$	s_2	$\{g_1,s_2\}$
$\{s_1\}$	g_2	$\{s_1,g_2\}$
$\{s_1\}$	s_2	$\{s_1,s_2\}$
$\{g_1,g_2\}$	p_1	$\{g_1,g_2,p_1\}$
$\{g_1,g_2\}$	p_2	$\{g_1,g_2,p_2\}$
$\{g_1,s_2\}$	p_1	$\{g_1,s_2,p_1\}$
$\{g_1,s_2\}$	p_2	$\{g_1,s_2,p_2\}$
$\{s_1,g_2\}$	p_1	$\{s_1,g_2,p_1\}$
$\{s_1,g_2\}$	p_2	$\{s_1,g_2,p_2\}$

TABLE 2.2. Tabular portrayal of the action tree (A,T) formally defined by $A = \{g_1,s_1,\ g_2,s_2,\ p_1,p_2\}$ and $T = \{\,\{\},\ \{g_1\},\ \{s_1\},\ \{g_1,g_2\},\ \{g_1,s_2\},\ \{s_1,g_2\},\ \{s_1,s_2\},\ \{g_1,g_2,p_1\},\ \{g_1,g_2,p_2\},\ \{g_1,s_2,p_1\},\ \{g_1,s_2,p_2\},\ \{s_1,g_2,p_1\},\ \{s_1,g_2,p_2\}\,\}$.

Let an *action tree* (A,T) be a nonempty finite set A and a collection $T \subseteq \mathcal{P}(A)$ such that $A = \bigcup T$ and such that

$$(1) \qquad \begin{gathered}(t,a) \mapsto t{\cup}\{a\} \text{ is an invertible function} \\ \text{from } F \text{ onto } T{\sim}\{\{\}\}\ .\end{gathered}$$

The assumption $A = \bigcup T$ holds without loss of generality because $A \supseteq \bigcup T$ by construction and because $A{\sim}\bigcup T$ can be made empty by eliminating unused actions.

More complicated is assumption (1). In Examples 1 and 2, this assumption corresponds to the fact that each noninitial node appears exactly once in the table's third column. In general, $(t,a) \mapsto t{\cup}\{a\}$ is always a function from the set F into $T{\sim}\{\{\}\}$. However, this unnamed function's invertibility is a substantial restriction: its surjectivity requires that every non-initial node has an immediate predecessor, and its injectivity requires that that immediate predecessor is unique. This assumption and the finiteness of A together imply that the N elements of any nonempty node t can be uniquely numbered as a_1, $a_2 \dots a_N$ in such a way that $a_1 \in F(\{\})$ and

$$(\forall n{\in}\{2,3,\dots N\})\ a_n \in F(\{a_1,a_2,\dots a_{n-1}\})$$

(Proposition A.1). This implies (among many other things) that every action tree must have $\{\}$ as a node.

In any action tree, the set of nodes T can be partitioned into the set of *terminal* nodes,

$$Z = \{\ z{\in}T \mid F(z){=}\varnothing\ \}\ ,$$

and the set of *nonterminal* nodes,

$$T{\sim}Z = \{\ t{\in}T \mid F(t){\neq}\varnothing\ \}\ .$$

Let an *action-based game form* (A,T,H) be an action tree (A,T) together with a family $H \subseteq \mathcal{P}(T{\sim}Z)$ of *agents* (i.e., information sets) h such that H partitions $T{\sim}Z$ and such that

(2a) $$(\forall t_1,t_2)\ \ [(\exists h)\{t_1,t_2\}{\subseteq}h]\ \Rightarrow\ F(t_1){=}F(t_2)\ \ \text{and}$$

(2b) $$(\forall t_1,t_2)\ \ [(\exists h)\{t_1,t_2\}{\subseteq}h]\ \Leftarrow\ F(t_1){\cap}F(t_2){\neq}\varnothing\ .$$

The first of the two implications states that the same actions are feasible from any two nodes in an agent. The second implication states that if two nodes share an action then they must share an agent. Both assumptions are standard.

3. Equivalence with a node-based game form

This section shows the equivalence between action-based game forms and node-based game forms having perfect agent recall. Dots distinguish node-based symbols from analogous symbols in the action-based formulation.

As is standard in the literature, let a *game tree* $(\dot{T},o,E)$ be a set $\dot{T}$ of *nodes* $\dot{t}$, an *initial node* $o \in \dot{T}$, and a set $E \subseteq \dot{T}^2$ of *edges* $(\dot{t}_1,\dot{t}_2)$ such that $(\forall(\dot{t}_1,\dot{t}_2){\in}E)\ \dot{t}_1{\neq}\dot{t}_2$ and such that

(3) $$\begin{gathered}(\forall\dot{t})(\exists!(\dot{t}_0,\dot{t}_1,\dots\dot{t}_N))\\ \dot{t}_0{=}o,\ \ \dot{t}_N{=}\dot{t},\ \ \text{and}\ (\forall n{\in}\{1,\dots N\})(\dot{t}_{n-1},\dot{t}_n){\in}E.\end{gathered}$$

This assumption states that every node $\dot{t}$ has a unique sequence of edges leading back to the initial node o. In the singular case $\dot{t}{=}o$, the sequence $(\dot{t}_0,\dot{t}_1,\dots\dot{t}_N)$ is $(\dot{t}_0){=}(\dot{t}_N){=}(o)$.

The above calls each $(\dot{t}_1,\dot{t}_2){\in}E$ an "edge," as in graph theory. Alternatively, we might interpret $(\dot{t}_1,\dot{t}_2){\in}E$ to mean that $\dot{t}_1$ "immediately preceeds" $\dot{t}_2$. From this alternative perspective, it is natural to regard E as a binary relation and to observe that its transitive closure is the precedence relation $\prec$ appearing in Kreps and Wilson (1982) and elsewhere.

Let a *labelled game tree* $(\dot{T},o,E,A,\alpha)$ be a game tree $(\dot{T},o,E)$ together with a set A of *actions* a and a surjective function $\alpha{:}E{\rightarrow}A$ such that

$$(\forall\dot{t})\text{ the restriction of }\alpha\text{ to }\{(\dot{t}_1,\dot{t}_2){\in}E|\dot{t}_1{=}\dot{t}\}\text{ is injective.} \tag{4}$$

Since $\{(\dot{t}_1,\dot{t}_2){\in}E|\dot{t}_1{=}\dot{t}\}$ is the set of edges leaving $\dot{t}$, this assumption requires that actions unambiguously label the edges leaving any node $\dot{t}$.

Let

$$\dot{F}(\dot{t}) = \alpha(\ \{(\dot{t}_1,\dot{t}_2){\in}E|\dot{t}_1{=}\dot{t}\}\ )$$

be the set of actions which label the edges leaving $\dot{t}$. As with F in the previous section, the symbol $\dot{F}$ was chosen to abbreviate "feasibility" because $\dot{F}(\dot{t})$ specifies the set of actions that are feasible from $\dot{t}$. In the literature, it is standard to let the symbol A denote not only the set of actions (as above) but also the feasibility correspondence (which is denoted here by $\dot{F}$). We need to make this distinction explicit, and accordingly, $\dot{F}(\dot{t})$ rather than $A(\dot{t})$ denotes the set of actions available from $\dot{t}$.

Next partition set of nodes $\dot{T}$ into the set of *terminal* nodes

$$\dot{Z} = \{\ \dot{z}{\in}\dot{T}\ |\ \dot{F}(\dot{z}){=}\varnothing\ \}\ ,$$

and the set of *nonterminal* nodes

$$\dot{T}{\sim}\dot{Z} = \{\ \dot{t}{\in}\dot{T}\ |\ \dot{F}(\dot{t}){\neq}\varnothing\ \}\ .$$

Finally, let a *node-based game form* $(\dot{T},o,E,A,\alpha,\dot{H})$ be a labelled game tree $(\dot{T},o,E,A,\alpha)$ together with a collection $\dot{H}\subseteq\mathcal{P}(\dot{T}{\sim}\dot{Z})$ of *agents* (i.e. information sets) $\dot{h}$ such that $\dot{H}$ is a partition of $\dot{T}{\sim}\dot{Z}$ and such that

$$\begin{array}{c}(\forall\dot{t}_1,\dot{t}_2)\ [(\exists\dot{h})\{\dot{t}_1,\dot{t}_2\}{\subseteq}\dot{h}]\Rightarrow\dot{F}(\dot{t}_1){=}\dot{F}(\dot{t}_2)\ \text{ and}\\ (\forall\dot{t}_1,\dot{t}_2)\ [(\exists\dot{h})\{\dot{t}_1,\dot{t}_2\}{\subseteq}\dot{h}]\Leftarrow\dot{F}(\dot{t}_1){\cap}\dot{F}(\dot{t}_2){\neq}\varnothing\ \ .\end{array} \tag{5}$$

As with (2a), the first of the two implications states that the same actions are feasible from any two nodes in an agent. And as with (2b), the second implication states that if two nodes share an action then they must share an agent. Both assumptions are standard.

3.1. Sufficiency of an action-based game form.

First we will show that every action-based game form determines a node-based game form. In addition, we will show that that node-based game form $(\dot{T}, o, E, A, \alpha, \dot{H})$ must have *perfect agent recall* in the sense that

$$(\nexists h, (\dot{t}_0, \dot{t}_1, ... \dot{t}_N))\ \{\dot{t}_0, \dot{t}_N\} \subseteq \dot{h} \text{ and } (\forall n \in \{1, ... N\})(\dot{t}_{n-1}, \dot{t}_n) \in E\ .$$

Perfect recall precludes the existence of a path which leaves an agent and later returns to the same agent.

Theorem 3.1. *Suppose (A, T, H) is an action-based game form. Then (T, o, E, A, α, H) is a node-based game form, where $o = \{\}$,*

$$E = \{\ (t_1, t_2) \mid (\exists a \notin t_1)\ t_1 \cup \{a\} = t_2\ \}\ ,\ \textit{and}$$
$$(\forall (t_1, t_2) \in E)\ \alpha(t_1, t_2)\ \textit{is the}\ a \notin t_1\ \textit{such that}\ t_1 \cup \{a\} = t_2\ .$$

Further, (T, o, E, A, α, H) has perfect agent recall. And finally, $F = \dot{F}$ and $Z = \dot{Z}$, where F and Z are derived from (A, T), and $\dot{F}$ and $\dot{Z}$ are derived from (T, o, E, A, α). (Proof in Appendix B.)

3.2. Necessity of an action-based game form.

Next we show the converse, namely, that every node-based game form having perfect agent recall determines an action-based game form. To express this result, define $S : \dot{T} \rightarrow \mathcal{P}(A)$ by

$$(\forall \dot{t})\ S(\dot{t}) = \alpha(\ \{(\dot{t}_{n-1}, \dot{t}_n) | n \geq 1\}\ )\ ,$$

where $(\dot{t}_0, \dot{t}_1, ... \dot{t}_N)$ specifies the unique sequence of edges leading from o to $\dot{t}$. Thus S maps a node $\dot{t}$ to the set $S(\dot{t})$ of actions leading to it. Then let

$$T = S(\dot{T})\ .$$

T is the range of the function S, that is, the collection of action sets that are each the image of some node in $\dot{T}$. Finally, let

$$H = \{\ S(\dot{h}) \mid \dot{h} \in \dot{H}\ \}\ .$$

Each agent $h{\in}H$ is, for some $\dot{h}{\in}\dot{H}$, the collection of action sets that are each the image of some node in $\dot{h}$. Thus each h is a subcollection of T just as each $\dot{h}$ is a subset of $\dot{T}$.

THEOREM 3.2. *Let $(\dot{T}, o, E, A, \alpha, \dot{H})$ be a node-based game form with perfect agent recall, and define $S{:}\dot{T}{\to}T$ and $H{\subseteq}\mathcal{P}(T)$ as above. Then S is invertible, (A, T, H) is an action-based game form, and*

$$(\forall \dot{t}, \dot{h})\ \dot{t}{\in}\dot{h} \Leftrightarrow S(\dot{t}){\in}S(\dot{h})\ . \tag{6}$$

Further,

$$\begin{aligned} &(\forall \dot{t}, a)\ (\dot{t}, a){\in}\dot{F} \Leftrightarrow (S(\dot{t}), a){\in}F\ , \\ &\textit{and } (\forall \dot{t})\ \dot{t}{\in}\dot{Z} \Leftrightarrow S(\dot{t}){\in}Z\ . \end{aligned} \tag{7}$$

where $\dot{F}$ and $\dot{Z}$ are derived from (T, o, E, A, α) and F and Z are derived from (A, T). (Proof in Appendix B.)

4. Defining an action-based game

A game is a node- or action-based game form together with players which group its agents, probabilities for the chance player, and payoffs for each strategic player at each of the terminal nodes. This section merely specifies these three additional components in our paper's notation.

Since all the nodes of an agent h have the same set of feasible actions by assumption (2a), it is natural to call this set the *agent's* set of feasible actions, and to denote it by $F(h)$. Formally, as with any correspondence, the value $F(h)$ of F at the set h is defined to be $\{a|(\exists t{\in}h)a{\in}F(t)\}$. This construction is particularly natural here because assumption (2a) implies that $(\forall t{\in}h)\ F(t) = F(h)$.

Further, $(F(h))_h$ is an indexed partition of A, that is, $\{F(h)|h\}$ partitions A and $h \mapsto F(h)$ is an invertible function. Proposition A.2 derives this fact from assumptions $A{=}\bigcup T$, (1), and (2).

A game will specify a partition P of the set of agents h into *players* p. Because the set of agents h partitions the set of nonterminal nodes, $(\bigcup_{h\in p} h)_p$ is an indexed (and likely coarser) partition of the set of nonterminal nodes. Further, because $(F(h))_h$ is an indexed partition of the set of actions by the previous paragraph, $(\bigcup_{h\in p} F(h))_p$ is an indexed (and likely coarser) partition of the set of actions. Thus we may unambiguously speak of "a player's agents," "a player's nodes," and "a player's actions:" p itself is the set of player p's agents, $\bigcup_{h\in p}h$ is the set of player p's nodes, and $\bigcup_{h\in p}F(h)$ is the set of player p's actions.

A game may or may not have a distinguished player $\tilde{p}$ known as the *chance player*. As with any other player, the chance player $\tilde{p}$ has its own set $\tilde{p}$ of agents, its own set $\bigcup_{h\in\tilde{p}}h$ of nodes, and its own set $\bigcup_{h\in\tilde{p}}F(h)$ of actions. If P does not contain $\tilde{p}$ (that is, if there is no chance agent), then both $\bigcup_{h\in\tilde{p}}h$ and $\bigcup_{h\in\tilde{p}}F(h)$ are empty. A *strategic player* is an element of $P{\sim}\{\tilde{p}\}$, that is, a non-chance player.

A *game* (A,T,H,P,ρ,u) is an action-based game form (A,T,H) together with (1) a collection $P\subseteq\mathcal{P}(H)$ of players p which partition H and which may or may not contain the distinguished player $\tilde{p}$, (2) a function $\rho:\bigcup_{h\in\tilde{p}}F(h)\rightarrow(0,1]$ which assigns a positive probability to each of the chance agent $\tilde{p}$'s actions, and (3) a function $u:(P{\sim}\{\tilde{p}\})\times Z \rightarrow\mathbb{R}$ which specifies a *payoff* $u_p(z)$ to each strategic player p at each terminal node z. The chance probabilities must satisfy $(\forall h{\in}\tilde{p})\ \Sigma_{a\in F(h)}\rho(a) = 1$ so that they specify a probability distribution for each chance agent $h{\in}\tilde{p}$. If P does not contain $\tilde{p}$, that is, if there is no chance player, then the function ρ is empty because $\cup_{h\in\tilde{p}}F(h)$ is empty.

Although beauty does remain in the eye of the beholder, some may find it pleasing that $P\subseteq\mathcal{P}(H)$, $H\subseteq\mathcal{P}(T)$, and $T\subseteq\mathcal{P}(A)$. We have built a tower of boxes containing boxes: players are sets of agents, agents are sets of nodes, and nodes are sets of actions.

Appendix A. Properties of action-based game forms

The text refers to two results which follow from the definition of an action-based game form. Here they are.

Proposition A.1. *Let (A,T) be an action tree, take any nonempty $t{\in}T$, let N be the number of elements (i.e., actions) in t, and define $t_N{=}t$. Then there exist a unique $(t_0,\ldots t_{N-1})$ such that*

$$(\forall n{\in}\{1,\ldots N\})\ (\exists a_n{\notin}t_{n-1})\ t_{n-1}{\cup}\{a_n\}{=}t_n\ .$$

Further, the accompanying $(a_1,\ldots a_N)$ is unique, $t_0{=}\varnothing$, and

$$(\forall n{\in}\{1,\ldots N\})\ t_n = \{a_1,a_2,\ldots a_n\}\ .$$

Proof. The invertibility of the function $(t,a)\mapsto t{\cup}\{a\}$ from F onto $T{\sim}\{\varnothing\}$ implies the following: For any nonempty t, there exists a unique

t_- for which

$$(\exists a{\notin}t_-)\ t_-{\cup}\{a\}{=}t\ ,$$

and further, the accompanying a is unique.

We now prove by backward induction on m that for all $m \in \{1, ... N\}$ there exists a unique $(t_{m-1}, ... t_{N-1})$ such that

$$(\forall n{\in}\{m, ... N\})\ (\exists a_n{\notin}t_{n-1})\ t_{n-1}{\cup}\{a_n\}{=}t_n\ ,$$

and further, the accompanying $(a_m, ... a_N)$ is unique, t_{m-1} has $m{-}1$ elements, and

$$(\forall n{\in}\{m, ... N\})\ t_{m-1}{\cup}\{a_m, ... a_n\}{=}t_n\ .$$

At the initial step of the induction ($m{=}N$), apply the preceding paragraph to t_N. At each successive backward step ($m{\in}\{1, ... N{-}1\}$), apply the preceding paragraph to t_m.

The conclusion of the last paragraph at $m{=}1$ reveals that there is a unique $(t_0, ... t_{N-1})$ such that

$$(\forall n{\in}\{1, ... N\})\ (\exists a_n{\notin}t_{n-1})\ t_{n-1}{\cup}\{a_n\}{=}t_n\ ,$$

and further, the accompanying $(a_1, ... a_N)$ is unique, t_0 has 0 elements, and

$$(\forall n{\in}\{1, ... N\})\ t_0{\cup}\{a_1, ... a_n\}{=}t_n$$

t_0 must be $\varnothing$ because it has zero elements. □

PROPOSITION A.2. *$\{F(h)\}_{h\in H}$ is an indexed partition of A. In other words, $\{F(h)|h\}$ partitions A and $h \mapsto F(h)$ is invertible.*

Proof. We begin with three observations.

(1) Each $F(h)$ is nonempty. This holds because each h is a subset of nonterminal nodes.

(2) If $h_1{\neq}h_2$ then $F(h_1){\cap}F(h_2) = \varnothing$. To see this, take any $h_1{\neq}h_2$, any $t_1{\in}h_1$, and any $t_2{\in}h_2$. Since H is a partition, we have $(\nexists h)\{t_1, t_2\}{\subseteq}h$, and hence $F(t_1){\cap}F(t_2) = \varnothing$ by the contrapositive of (2b). This implies $F(h_1){\cap}F(h_2){=}\varnothing$ because $F(t_1){=}F(h_1)$ by $t_1{\in}h_1$ and (2a), and because $F(t_2){=}F(h_2)$ by $t_2{\in}h_2$ and (2a).

(3) $\bigcup\{F(h)|h\} = A$. $\bigcup\{F(h)|h\}{\subseteq}A$ follows from the definition of F. To see the converse, take any a. By the assumption $A{=}\bigcup T$, there exists some t such that $a{\in}t$. By Proposition A.1, there exists some t_{n-1} and t_n such that $t_{n-1}{\cup}\{a\}{=}t_n$. Hence $a{\in}F(t_{n-1})$ by the definition of F. Finally, since H partitions the collection of nonterminal nodes and

t_{n-1} is nonterminal, there is some h containing t_{n-1}. So $a{\in}F(h)$ by the last two sentences.

$\{F(h)|h\}$ partitions A by observations (1)-(3). If $h \mapsto F(h)$ were not invertible, there would be $h_1{\neq}h_2$ such that $F(h_1){=}F(h_2)$. Since both $F(h_1)$ and $F(h_2)$ are both nonempty by observation (1), we would then have $h_1{\neq}h_2$ such that $F(h_1){\cap}F(h_2){\neq}\varnothing$. This would contradiction observation (2). □

Appendix B. Proofs of theorems

B.1. Sufficiency of an action-based game form.

Proof of Theorem 3.1. Let (A,T,H) be an action-based game form and define o, E, and α as in the proposition's statement.

(T,o,E) is a game tree. The definition of E yields that $(\forall(t_1,t_2){\in}E)$ $t_1{\neq}t_2$. To prove (3), take any t. By Lemma A.1 there exist unique $(t_0,\dots t_N)$ such that $t_0=\varnothing$, $t_N=t$, and

$$(\forall n{\in}\{1,\dots N\})\ (\exists a_n{\notin}t_{n-1})\ t_{n-1}{\cup}\{a_n\}{=}t_n\ .$$

By the definition of o we have $t_0=o$, and by the definition of E we have $(\forall n{\in}\{1,\dots N\}\ (t_{n-1},t_n){\in}E$.

$(T,\varnothing,E,A,\alpha)$ is a labelled game tree. To see that $\alpha{:}E{\rightarrow}A$ is surjective, take any a. Since $\{F(h)|h\}$ partitions A by Lemma A.2, there is some h such that $a{\in}F(h)$. Take any $t{\in}h$. Since $F(t){=}F(h)$ by (2a), $a{\in}F(t)$. Hence we have $a{\notin}t$ and $t{\cup}\{a\}{\in}T$ by the definition of F, and therefore $a{=}\alpha(t,t{\cup}\{a\})$ by the definition of α.

To prove (4), take any t and any distinct (t,t^1) and (t,t^2) in E. By the definition of E, there exist distinct a^1 and a^2 such that $a^1{\notin}t$ and $t{\cup}\{a^1\}{=}t^1$ and such that $a^2{\notin}t$ and $t{\cup}\{a^2\}{=}t^2$. Thus $\alpha(t,t^1){=}a^1$ is distinct from $\alpha(t,t^2){=}a^2$ by the definition of α.

$(T,\varnothing,E,A,\alpha,H)$ is a node-based game form. Derive $\dot{F}$ and $\dot{Z}$ from the labelled game tree $(T,\varnothing,E,A,\alpha)$. By the definitions of $\dot{F}$, E, α, and F,

$$\begin{aligned}
\dot{F}(t) &= \alpha(\ \{(t,t_+){\in}E\}\ ) \\
&= \{\ \alpha(t,t_+)\ |\ (t,t_+){\in}E\ \} \\
&= \{\ \alpha(t,t_+)\ |\ (\exists a{\notin}t)\ t{\cup}\{a\}{=}t_+\ \} \\
&= \{\ a\ |\ (\exists t_+)\ a{=}\alpha(t,t_+) \text{ and } (\exists a{\notin}t)\ t{\cup}\{a\}{=}t_+\ \} \\
&= \{\ a\ |\ (\exists t_+)\ a{\notin}t \text{ and } t{\cup}\{a\}{=}t_+\ \}
\end{aligned}$$

$$= \{\ a{\notin}t \mid t{\cup}\{a\}{\in}T\ \}$$
$$= F(t)\ .$$

Hence $\dot{Z} = \{t|\dot{F}(t){=}\varnothing\} = \{t|F(t){=}\varnothing\} = Z$. Because $\dot{F}{=}F$ and $\dot{Z}{=}Z$, (5) is equivalent to (2).

Perfect Recall. If perfect agent recall were violated, there would be an h and a $(t_0, t_1, ...t_N)$ such that

$$\{t_0, t_N\}{\subseteq}h \text{ and } (\forall n{\in}\{1, ...\, N\})(t_{n-1}, t_n){\in}E\ .$$

Let $a_0 = \alpha(t_0, t_1)$. By $(\forall n{\in}\{1, ...\, N\})$ $(t_{n-1}, t_n){\in}E$ and the definition of E, we have $a_0 \in t_N$, and thus by the definition of F, we have $a_0 \notin F(t_N)$. And yet, by $a_0 \in F(t_0)$, $\{t_0, t_N\}{\subseteq}h$, and (2b), we have $a_0 \in F(t_N)$. □

B.2. Necessity of an action-based game form.

Proof of Theorem 3.2. Invertibility. S is surjective because T was defined to be its range. If S were not injective, there would be $\dot{t}_1$ and $\dot{t}_2$ such that $\dot{t}_1{\neq}\dot{t}_2$ and yet $S(\dot{t}_1){=}S(\dot{t}_2)$. Let $\dot{t}_0$ be the node at which the path from o to $\dot{t}_1$ diverges from the path to $\dot{t}_2$. Then let a_1 label the edge leaving $\dot{t}_0$ toward $\dot{t}_1$ and let a_2 label the edge leaving $\dot{t}_0$ toward $\dot{t}_2$.

Note that $a_2{\neq}a_1$ because the two edges leaving $\dot{t}_0$ are distinct, and because α is assumed to be injective on the edges leaving any node. Also note that $a_2{\in}S(\dot{t}_1)$ because $S(\dot{t}_1){=}S(\dot{t}_2)$ by assumption, and because $a_2{\in}S(\dot{t}_2)$ because it labels an edge leading to $\dot{t}_2$.

Because $a_2{\neq}a_1$ and $a_2{\in}S\{\dot{t}_1\}$, there must be some $\dot{t}_{00}{\neq}\dot{t}_0$ on the path to $\dot{t}_1$ such that $a_2{\in}\dot{F}(\dot{t}_{00})$. By its definition, a_2 also belongs to $\dot{F}(\dot{t}_0)$, and thus by (5b), $\dot{t}_0$ and $\dot{t}_{00}$ belong to the same agent. Since the path from o to $\dot{t}_1$ passes through both $\dot{t}_0$ and $\dot{t}_{00}$, it must pass through their common agent twice, in violation of perfect agent recall.

An Intermediate Step. The next four paragraph show

$$\begin{aligned}(8)\qquad & (\forall \dot{t}, \dot{t}_+, a)\ [\ (\dot{t}, \dot{t}_+){\in}E \text{ and } a{=}\alpha(\dot{t}, \dot{t}_+)\ ] \\ & \quad\Leftrightarrow [\ a{\notin}S(\dot{t}) \text{ and } S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)\ ]\ .\end{aligned}$$

The easier half is $\Rightarrow$. Assume $(\dot{t}, \dot{t}_+){\in}E$ and $a{=}\alpha(\dot{t}, \dot{t}_+)$. Let

$$(\dot{t}_0, \dot{t}_1, \dot{t}_2, ...\ \dot{t}_N) = (o, \dot{t}_1, \dot{t}_2, ...\ \dot{t})$$

denote the unique path from o to $\dot{t}$. Since $(\dot{t},\dot{t}_+){\in}E$,

$$(\dot{t}_0,\dot{t}_1,\dot{t}_2,\ldots\,\dot{t}_N,\dot{t}_{N+1}) = (o,\dot{t}_1,\dot{t}_2,\ldots\,\dot{t},\dot{t}_+)$$

specifies the unique path from o to $\dot{t}_+$. Thus

$$S(\dot{t}) = \{\ \alpha(o,\dot{t}_1),\ \alpha(\dot{t}_1,\dot{t}_2),\ \ldots\ \alpha(\dot{t}_{N-1},\dot{t})\ \}\text{ and}$$
$$S(\dot{t}_+) = S(\dot{t}) \cup \{\alpha(\dot{t},\dot{t}_+)\}\ .$$

Hence $a{=}\alpha(\dot{t},\dot{t}_+)$ yields $S(\dot{t}_+) = S(\dot{t}){\cup}\{a\}$, and further, $a{\notin}S(\dot{t})$ because $S(\dot{t}){\neq}S(\dot{t}_+)$, which follows from the invertibility of S and the fact that $\dot{t}{\neq}\dot{t}_+$ because $(\dot{t},\dot{t}_+){\in}E$.

The trickier half is $\Leftarrow$. To begin, this paragraph shows by contradiction that $S(\dot{t}_1) \subseteq S(\dot{t}_2)$ implies that $\dot{t}_1$ is on the path to $\dot{t}_2$. Accordingly, suppose that $S(\dot{t}_1) \subseteq S(\dot{t}_2)$ and yet $\dot{t}_1$ is not on the path to $\dot{t}_2$. Then there would be some $\dot{t}_0$ at which the path to $\dot{t}_1$ diverges from the path to $\dot{t}_2$. Let a_1 label the edge leaving $\dot{t}_0$ toward $\dot{t}_1$. Since a_1 is also in $S(\dot{t}_2)$ because $S(\dot{t}_1){\subseteq}S(\dot{t}_2)$, there must be a $\dot{t}_{00}$, other than $\dot{t}_0$, and yet on the path to $\dot{t}_2$, from which a_1 can be chosen. Since a_1 can be chosen from both $\dot{t}_0$ and $\dot{t}_{00}$, (5b) implies that these two nodes are in the same agent. Then since both are on the path to $\dot{t}_2$, the path to $\dot{t}_2$ must enter their agent twice, in contradiction to perfect agent recall.

Now assume that $a{\notin}S(\dot{t})$ and $S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)$. Since $S(\dot{t}){\subseteq}S(\dot{t}_+)$, the previous paragraph implies that $\dot{t}$ is on the path to $\dot{t}_+$. Now let $\dot{t}_1$ be the first node on the path from $\dot{t}$ to $\dot{t}_+$. For future reference, note

$$(\dot{t},\dot{t}_1) \in E\ . \tag{9}$$

Further, note

$$\alpha(\dot{t},\dot{t}_1) = a \tag{10}$$

because $\alpha(\dot{t},\dot{t}_1)$ cannot be inside $S(\dot{t})$ without making $S(\dot{t}_1){=}S(\dot{t})$ in violation of the invertibility of S, and it cannot be outside $S(\dot{t}){\cup}\{a\}$ without violating either $S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)$ or the fact that $\dot{t}_1$ is on the path to $\dot{t}_+$.

This paragraph proves by contradiction that there cannot be a second node on the path from $\dot{t}$ to $\dot{t}_+$. Accordingly, suppose $\dot{t}_2$ were such a second node. Unfortunately, $\alpha(\dot{t}_1,\dot{t}_2)$ cannot be inside $S(\dot{t}){\cup}\{a\}$ without making $S(\dot{t}_1) = S(\dot{t}_2)$ in violation of the invertibility of S, and, it cannot be outside $S(\dot{t}){\cup}\{a\}$ without either violating $S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)$

or the fact that $\dot{t}_2$ is on the path to $\dot{t}_+$. Thus $(\dot{t}_1, \dot{t}_2)$ cannot be labelled, which contradicts the definition of a labelled traditional game tree.

Since there cannot be a second node on the path from $\dot{t}$ to $\dot{t}_+$, it must be that $\dot{t}_+ = \dot{t}_1$. Hence (9) and (10) provide the required results.

(A,T) is an action tree. $A \supseteq \bigcup T$ by construction, and $A{\sim}\bigcup T = \varnothing$ by the surjectivity of α.

More subtly, we must show that $(t,a) \mapsto t{\cup}\{a\}$ is an invertible function from F onto $T{\sim}\{\varnothing\}$. Since F is defined to be

$$\{\ (t,a) \mid a{\notin}t \text{ and } t{\cup}\{a\}{\in}T\ \}\ ,$$

this is equivalent to showing

$$(\forall t_+{\in}T{\sim}\{\varnothing\})\ (\exists!(t,a))\ a{\notin}t \text{ and } t{\cup}\{a\}{=}t_+\ .$$

Because $S{:}\dot{T}{\rightarrow}T$ is invertible and $S(o){=}\varnothing$, this is equivalent to showing

$$(\forall \dot{t}_+{\in}\dot{T}{\sim}\{o\})\ (\exists!(\dot{t},a))\ a{\notin}S(\dot{t}) \text{ and } S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)\ .$$

Accordingly, take any $\dot{t}_+{\neq}o$. By the definition of a traditional game tree, there is a unique path leading from $\dot{t}_+$ back to o, and thus since $\dot{t}_+{\neq}o$, there is a unique $\dot{t}$ such that $(\dot{t}, \dot{t}_+){\in}E$. Hence, there is a unique $(\dot{t}, a)$ such that

$$(\dot{t}, \dot{t}_+){\in}E \text{ and } a = \alpha(\dot{t}, \dot{t}_+)\ .$$

Therefore by (8) and the invertibility of S, there is a unique $(\dot{t}, a)$ such that $a{\notin}S(\dot{t})$ and $S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+)$.

Derivation of (6) and (7). By the definition of H in the theorem statement and by the invertibility of S,

$$(\forall \dot{t}, \dot{h})\ \dot{t}{\in}\dot{h} \Leftrightarrow S(\dot{t}){\in}S(\dot{h})\ .$$

By the definition of $\dot{F}$, by (8), and by the definition of T as $S(\dot{T})$,

$$\begin{aligned}
&(\forall \dot{t}, a)\ \ (\dot{t}, a) \in \dot{F} \\
\Leftrightarrow\ \ &(\exists \dot{t}_+)\ (\dot{t}, \dot{t}_+){\in}E \text{ and } a{=}\alpha(\dot{t}, \dot{t}_+) \\
\Leftrightarrow\ \ &(\exists \dot{t}_+)\ a{\notin}S(\dot{t}) \text{ and } S(\dot{t}){\cup}\{a\}{=}S(\dot{t}_+) \\
\Leftrightarrow\ \ &a{\notin}S(\dot{t}) \text{ and } S(\dot{t}){\cup}\{a\}{\in}T \\
\Leftrightarrow\ \ &(S(\dot{t}), a) \in F\ .
\end{aligned}$$

Finally, the definition of $\dot{Z}$, the previous equivalence, and the definition of Z imply

$$(\forall \dot{t}) \ \ \dot{t} \in \dot{Z} \Leftrightarrow \dot{F}(\dot{t}) = \varnothing \Leftrightarrow F(S(\dot{t})) = \varnothing \Leftrightarrow S(\dot{t}) \in Z \ .$$

(A, T, H) *is a game form.* We have seen that (A, T) is an action tree. (6), (7), and the invertibility of S imply that (5) is equivalent to (2). $\square$

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