Monetary Policy and Distribution

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Abstract

A monetary model of heterogeneous households is constructed which deals in a tractable way with the distribution of money balances across the population. Only some households are on the receiving end of a money injection from the central bank, and this in general produces price dispersion across markets. This price dispersion generates uninsured consumption risk which is important in determining the effects of money growth, optimal policy, and the effects of money growth shocks. The optimal money growth rate can be very close to zero, and the welfare cost of small inflations can be very large. Small money shocks can have important sectoral effects with small effects on aggregates, while large money shocks can have proportionately large effects on aggregates.
1. INTRODUCTION

The purpose of this paper is to construct a tractable model that takes seriously the idea that the distributional effects of monetary policy are important for macroeconomic activity. We explore the qualitative and quantitative implications of this model for the effects of monetary policy on prices, output, consumption, and employment.

Models with distributional effects of monetary policy are certainly not new. The first models of this type were the limited participation models constructed by Grossman and Weiss (1983) and Rotemberg (1984), in which there are always some economic agents who are not participating in financial markets and will not be on the receiving end of an open market operation, at least on the first round. In a limited participation model, a monetary injection by the central bank causes a redistribution of wealth which will in general cause short run changes in asset prices, employment, output, and the distribution of consumption across the population. The subsequent literature has to a large extent focussed on asset pricing implications, particularly Lucas (1990), Alvarez and Atkeson (1997), and Alvarez, Atkeson, and Kehoe (2002), in models that, for tractability, finesse some of the potentially interesting distributional implications of monetary policy.

Recent research in monetary theory is aimed at developing models of monetary economies that capture heterogeneity and the distribution of wealth in a manner that is tractable for analytical and quantitative work. One approach is to use a quasi-linear utility function as in Lagos and Wright (2005), which under some circumstances will lead to economic agents optimally redistributing money balances uniformly among themselves when they have the opportunity. Another approach is to use a representative household with many agents, as in Shi (1997), in which (also see Lucas 1990) there can be redistributions of wealth within the household during the period, but these distribution effects do not persist. Work by Williamson (2005) and Shi (2004)
uses the quasi-linear-utility and representative-household approaches, respectively, to study some implications of limited participation for optimal monetary policy, interest rates, and output. Other related work is Head and Shi (2003) and Chiu (2004).

In the model constructed in this paper, the existence of many-agent households aids in allowing us to deal with the distribution of wealth in a tractable way, but there is sufficient heterogeneity among households to permit some novel implications for monetary policy. There is only one asset, fiat money, and the central bank intervenes by making money transfers to households. These transfers are received by some households, and not by others. A key feature of the model is that, in each period, exchange occurs between members of households who received the transfer and members of households who do not. That is, households are spatially separated, and each period the agents from a household who purchase goods are dispersed to other locations. In this way, a money injection by the central bank diffuses through the economy over time, and in the limit there will be no distributional effect of monetary policy.

In equilibrium, prices are in general different across locations. Since individual agents are uncertain about where they will be buying goods, and there are no markets on which to insure this risk, if there is dispersion in prices then this is an important source of uncertainty for the household. With constant money growth, there will in general be permanent price dispersion. From the point of view of the policymaker, there are two key distortions. The first is the standard monetary distortion - because agents discount the future, with constant prices they will tend to hold too little real money balances, and this distortion can be corrected through a deflation that gives money an appropriately high real rate of return. The second is the relative price distortion - in the model, if the money supply is growing or shrinking then price dispersion exists and agents face consumption risk. The second distortion can be corrected if the money supply is constant, which implies constant prices in the
model. The optimal money growth rate is therefore negative, but larger than minus the rate of time preference. That is, a Friedman rule is not optimal here. This is a key result, as the Friedman rule is probably the most ubiquitous of properties of monetary models. Some numerical experiments show that there are circumstances where the optimal money growth rate is very close to zero, so that the welfare loss from price stability is extremely small. The welfare loss from a moderate inflation of course depends on parameter values, but we show examples of moderate inflations that yield welfare costs of inflation that are an order of magnitude higher than those typically found in the literature, even with levels of risk aversion that are moderate.

To illustrate the dynamic effects of central bank money injections, we study a stochastic version of the model. Even i.i.d. money shocks yield persistent effects on output and employment, and this persistence depends critically on a parameter that governs the speed of diffusion of money through the economy. Numerical examples show that monetary shocks can be quantitatively important, particularly for the distribution of consumption across the population, and for sectoral employment. Large effects on real aggregates are also possible, if money shocks are large.

In Section 2 we construct the model, while in Sections 3 and 4 we study the effects of constant money growth and stochastic money growth, respectively. Section 5 is a conclusion.

2. THE MODEL

There is a continuum of islands with unit mass indexed by \(i \in [0, 1]\). Each island has a double infinity of locations indexed by \(j = -\infty, ..., -1, 0, 1, ..., \infty\). At each location there is an infinitely-lived household, consisting of a producer and a continuum of consumers, with the continuum of consumers having unit mass. Consumers are indexed by \(k \in [0, 1]\). The preferences of a household at location \(j\) on island \(i\) are
given by
\[ E_0 \sum_{t=0}^{\infty} \beta^t \left[ \int_0^1 u(c_t^{ij}(k))d\lambda(k) - v(n_t^{ij}) \right], \tag{1} \]

where \( t \) indexes time, \( 0 < \beta < 1 \), \( c_t^{ij}(k) \) is the consumption of consumer \( k \) who is a member of the household living at location \( j \) on island \( i \), \( n_t^{ij} \) is the labor supply of the producer who is a member of the household living at location \( j \) on island \( i \), and \( \lambda(\cdot) \) denotes the measure of consumers in the household. Assume that \( u(\cdot) \) is twice continuously differentiable and strictly concave, with \( u'(0) = \infty \). Also suppose that \( v(\cdot) \) is twice continuously differentiable and strictly convex, with \( v'(0) = 0 \) and \( v'(\infty) = \infty \). The producer can supply an unlimited quantity of labor, and each unit of labor supplied yields one unit of the perishable consumption good.

There is a fraction \( \alpha \) of connected islands, where \( 0 < \alpha < 1 \). At the beginning of the period, the household at location \( j \) on island \( i \) has \( m_t^{ij} \) units of divisible fiat money. All of the households on connected islands then receive an identical money transfer \( \Upsilon_t \) from the central bank. Households on islands that are not connected never receive transfers. After receiving transfers, the household must decide how much money to allocate to each of its consumers. After receiving money from the household, each consumer receives a location shock. There is a probability \( \pi \) that a consumer stays on the same island, and a probability \( 1 - \pi \) that the consumer is randomly relocated to another island. On each island there is an absence-of-double-coincidence problem. That is, consumers from a household \( j \) desire the consumption good produced by household \( j + 1 \). The same applies to consumers who change islands. That is, a consumer from location \( j \) on his or her home island desires the consumption goods produced by the producer at location \( j + 1 \) at the island to which he or she is relocated.

While consumers go shopping for goods, producers remain at home and sell goods to consumers arriving from other locations. When consumers purchase consumption goods, these goods must be consumed on the spot, and the consumers then return to
their home locations. Thus, the household’s consumers cannot share risk by returning to their home location and pooling their consumption goods. Assuming that there is no communication among locations and no record-keeping, consumption goods must be purchased with money. Thus, consumers face cash-in-advance constraints.

The key features of the model are that goods must be purchased with money, that money injections and withdrawals by the central bank will alter the distribution of money balances across the population, and consumption goods cannot be moved between locations.

3. CONSTANT MONEY GROWTH

In this section, we will first characterize an equilibrium where the money stock grows at a constant rate. Then, we determine the effects of changes in the money growth rate on employment, output, consumption and prices across locations. Next, we will draw some general conclusions about the optimal money growth rate, followed by some numerical examples.

Consumers from a given household who find themselves at different locations will in general face different prices for consumption goods. In the equilibria we study, prices will be identical on all connected islands and on all unconnected islands. Then, let $p_{1t}$ ($p_{2t}$) denote the price of goods in terms of money on connected (unconnected) islands. If the household knew where consumers were to be located when it makes its decision about how to distribute household money balances among consumers, it would in general want to give different agents different money allocations. However, since consumers’ location shocks are unknown when this decision is made, it is optimal for the household to distribute money balances equally among consumers.

A household on a connected island has $m_{1t}$ units of money balances at the beginning of period $t$, and receives a nominal transfer $\Upsilon_t$, so the household optimally allocates $m_{1t} + \Upsilon_t$ units of money to each consumer. Then relocation shocks are realized and
$1 - (1 - \alpha)\pi$ consumers go to connected islands, with each consuming $c_{1t}^{11}$ goods which are purchased at the price $p_{1t}$.

The consumers that travel from a connected island to a connected island face the cash-in-advance constraint

$$p_{1t}c_{t}^{11} \leq m_{1t} + \Upsilon_{t}.$$  \hspace{1cm} (2)

As well, $(1 - \alpha)\pi$ consumers go to unconnected islands and consume $c_{t}^{12}$ goods purchased at the price $p_{2t}$, facing the cash-in-advance constraint

$$p_{2t}c_{t}^{12} \leq m_{1t} + \Upsilon_{t}.$$  \hspace{1cm} (3)

The producer remains at the home location, supplying $n_{1t}$ units of labor to produce $n_{1t}$ consumption goods, which are then sold at the price $p_{1t}$. Consumers return with any unspent money and add this to the money acquired through the sale of goods, and the household enters period $t + 1$ with $m_{1,t+1}$ units of money. The household’s budget constraint is then

$$[1 - (1 - \alpha)\pi]p_{1t}c_{t}^{11} + (1 - \alpha)\pi p_{2t}c_{t}^{12} + m_{1,t+1} = p_{1t}n_{1t} + m_{1t} + \Upsilon_{t}.$$  \hspace{1cm} (4)

Similarly, a household on a unconnected island begins period $t$ with $m_{2t}$ units of money which is allocated equally among the household’s consumers. After receiving location shocks, $\alpha\pi$ consumers from the household each arrive at connected islands and consume $c_{t}^{21}$ consumption goods each, facing the cash-in-advance constraint

$$p_{1t}c_{t}^{21} \leq m_{2t}.$$  \hspace{1cm} (5)

As well, $1 - \alpha\pi$ consumers travel to unconnected islands, with each consuming $c_{t}^{22}$ and facing the cash-in-advance constraint

$$p_{2t}c_{t}^{22} \leq m_{2t}.$$  \hspace{1cm} (6)

For a household on a unconnected island, the budget constraint is

$$\alpha\pi p_{1t}c_{t}^{21} + (1 - \alpha\pi) p_{2t}c_{t}^{22} + m_{2,t+1} = p_{2t}n_{2t} + m_{2t},$$  \hspace{1cm} (7)
or total nominal consumption expenditure by the household plus end-of-period money balances is equal to total receipts from sales of goods plus beginning-of-period money balances.

From (1) and (2)-(7), the first-order conditions for an optimum for connected and unconnected households, respectively, give

\[ -v'(n_{1t}) + \beta \left\{ \frac{p_{1t}(1 - (1 - \alpha)\pi)u'(c_{1,t+1}^1)}{p_{1,t+1}} + \frac{p_{1t}(1 - \alpha)\pi u'(c_{1,t+1}^2)}{p_{2,t+1}} \right\} = 0, \]  

\[ -v'(n_{2t}) + \beta \left\{ \frac{p_{2t}\alpha\pi u'(c_{2,t+1}^1)}{p_{1,t+1}} + \frac{p_{2t}(1 - \alpha\pi)u'(c_{2,t+1}^2)}{p_{2,t+1}} \right\} = 0. \]  

That is, each household supplies labor each period to produce consumption goods, which it sells for money. The money is then spent in the following period for consumption goods at connected and unconnected islands. Thus, in equations (8) and (9) at the optimum each household equates the current marginal disutility of labor with the discounted expectation of the gross real rate of return on money weighted by the marginal utility of consumption in the forthcoming period.

In all of the equilibria we examine, the cash-in-advance constraints (2), (3), (6), and (5) hold with equality. Then, since households always spend all of their money on consumption goods, the path for the money stock at each location is exogenous. In period \( t \), let \( M_{1t} \) denote the supply of money per household on each connected island after the transfer from the central bank, and let \( M_{2t} \) denote the supply of money per household on each unconnected island. At a location on a connected island, during period \( t \) there will be a total of \( 1 - (1 - \alpha)\pi \) agents who will arrive from connected islands, and each these agents will spend \( M_{1t} \) units of money in exchange for goods, while \( (1 - \alpha)\pi \) agents will arrive from unconnected islands and will spend \( M_{2t} \) units of money each. Similarly, at a location on an unconnected island, \( 1 - \alpha\pi \) agents will arrive from other unconnected islands, with each of these agents spending \( M_{2t} \) units of money, and \( M_{1t} \) units of money will be spent by each of the \( \alpha\pi \) agents arriving from connected islands. Therefore, the stocks of money per household at connected
and unconnected locations evolve according to

\[ M_{1,t+1}^1 = [1 - (1 - \alpha)\pi] M_{1t} + (1 - \alpha)\pi M_{2t} + \Upsilon_{t+1}, \quad (10) \]

\[ M_{2,t+1}^1 = \alpha \pi M_{1t} + (1 - \alpha \pi) M_{2t}. \quad (11) \]

Note that \( \pi \) will govern how quickly a money injection by the central bank becomes diffused through the economy. If \( \pi = 1 \), in which case all consumers are relocated to another island, then from (10) and (11) the same quantity of money is spent in all locations in each period, so that diffusion occurs in one period. If \( \pi = 0 \) then \( M_{2t} = M_{20} \) for all \( t \) and \( M_{1t} \) is governed entirely by the history of central bank transfers and does not depend on \( \alpha \).

The equilibrium conditions are

\[ p_{1t} n_{1t} = [1 - (1 - \alpha)\pi] M_{1t} + (1 - \alpha)\pi M_{2t}, \quad (12) \]

\[ p_{2t} n_{2t} = \alpha \pi M_{1t} + (1 - \alpha \pi) M_{2t}, \quad (13) \]

or money demand equals money supply on connected and unconnected islands, respectively. Then, substituting in (8) and (9) for consumption and prices using (2), (3), (6), and (5) (with equality in all four cases), (12), and (13), we get

\[ -v'(n_{1t}) n_{1t} \frac{[1 - (1 - \alpha)\pi] M_{1t+1}^1 + (1 - \alpha)\pi M_{2t+1}^1}{[1 - (1 - \alpha)\pi] M_{1t} + (1 - \alpha)\pi M_{2t}} + \beta \left\{ \frac{[1 - (1 - \alpha)\pi] \left[ M_{1,t+1}^1 \left( \frac{M_{1,t+1}^1}{M_{1,t+1}^1 + (1 - \alpha)\pi M_{2,t+1}^1} \right) n_{1,t+1} \right]}{[1 - (1 - \alpha)\pi] M_{1,t+1} + (1 - \alpha)\pi M_{2,t+1}} + \frac{(1 - \alpha)\pi n_{2,t+1}}{\alpha \pi M_{1,t+1} + (1 - \alpha \pi) M_{2,t+1}} \right\} = 0, \quad (14) \]

\[ -v'(n_{2t}) n_{2t} \frac{\alpha \pi M_{1t} + (1 - \alpha \pi) M_{2t}}{\alpha \pi M_{1t} + (1 - \alpha \pi) M_{2t}} + \beta \left\{ \frac{(1 - \alpha)\pi \left[ n_{2,t+1} \left( \frac{M_{2,t+1}^1}{M_{1,t+1}^1 + (1 - \alpha)\pi M_{2,t+1}^1} \right) n_{2,t+1} \right]}{(1 - \alpha)\pi M_{1,t+1} + (1 - \alpha \pi) M_{2,t+1}} + \frac{\alpha \pi M_{1,t+1}^1 + (1 - \alpha \pi) M_{2,t+1}^1}{\alpha \pi M_{1,t+1}^1 + (1 - \alpha \pi) M_{2,t+1}^1} \right\} = 0. \quad (15) \]
An equilibrium is then a sequence \( \{n_{1t}, n_{2t}\}_{t=0}^{\infty} \) that solves (14) and (15) for \( t = 0, 1, 2, \ldots \), with \( \{M_{1t}, M_{2t}\}_{t=0}^{\infty} \) determined by (10) and (11), given \( M_{10}, M_{20}, \) and \( \{\Upsilon_t\}_{t=0}^{\infty} \). Then equilibrium prices can be determined from (12) and (13), and consumption quantities are determined by

\[
\begin{align*}
c_{11}^t &= n_{1t} \frac{M_{1t}}{[1 - (1 - \alpha)\pi]M_{1t} + (1 - \alpha)\pi M_{2t}}, \quad (16) \\
c_{12}^t &= n_{2t} \frac{M_{1t}}{\alpha \pi M_{1t} + (1 - \alpha \pi)M_{2t}}, \quad (17) \\
c_{22}^t &= n_{2t} \frac{M_{2t}}{\alpha \pi M_{1t} + (1 - \alpha \pi)M_{2t}}, \quad (18) \\
c_{21}^t &= n_{1t} \frac{M_{2t}}{[1 - (1 - \alpha)\pi]M_{1t} + (1 - \alpha)\pi M_{2t}}, \quad (19)
\end{align*}
\]

**Money Growth**

Since the distribution of money balances across islands matters in this model, not every monetary policy rule with a constant growth rate for the aggregate money stock will yield an equilibrium that is straightforward to analyze. From (14) and (15) a money growth policy that will yield an equilibrium where \( n_{it} \) is constant for all \( t \), for \( i = 1, 2 \), is one where \( M_{it} \) grows at a constant gross rate \( \mu \) for each \( i = 1, 2 \). Given this policy, if we define \( \delta \) to be the ratio of the per-capita money stocks on connected and unconnected islands; that is

\[
\delta \equiv \frac{M_{1t}}{M_{2t}}, \quad (20)
\]

then clearly, \( \delta \) must be constant for all \( t \). Therefore, from (10) and (11), and again assuming binding cash-in-advance constraints, we must have

\[
\delta = \frac{M_{1,t+1}}{M_{2,t+1}} = \frac{\mu M_{1t}}{\alpha \pi M_{1t} + (1 - \alpha \pi)M_{2t}},
\]

which then requires

\[
\delta = \frac{\mu - 1 + \alpha \pi}{\alpha \pi}. \quad (21)
\]
That is, to implement a monetary policy where the aggregate money stock and per-capita money stocks in all locations grow at the same gross rate $\mu$, the monetary authority must set the transfer in period 0 so that the ratio of per-capita money stocks on connected and unconnected islands conforms to (21), and then transfers are made in each succeeding period so that the money stock on connected islands grows at the gross rate $\mu$. Thus, if the money supply growth rate is positive ($\mu > 1$) then from (21) there will be a higher quantity of money per capita at each date on connected islands than on unconnected islands, and vice-versa if $\mu < 1$.

Given this constant money growth policy, there exists an equilibrium where labor supply, output, and consumption for each type of consumer are constant for all time in each location. Letting $n_1$ and $n_2$ denote labor supply by a household on a connected and unconnected island, respectively then, from (14) and (15), $n_1$ and $n_2$ are determined by

\[-v'(n_1)n_1 + \beta \left\{ \left. \left[ 1 - (1 - \alpha)\pi \right] u' \right| n_1 \left( 1 - (1 - \alpha)\pi \right) + \alpha \pi \right\} \left( \frac{n_1}{\mu} \right) \left( \frac{\mu - 1 + \alpha \pi}{\alpha \pi} \right) = 0, \tag{22} \]

\[-v'(n_2)n_2 + \beta \left\{ \left. \left[ 1 - (1 - \alpha)\pi \right] u' \right| n_2 \left( 1 - (1 - \alpha)\pi \right) + \alpha \pi \right\} \left( \frac{n_2}{\mu} \right) \left( \frac{\mu - 1 + \alpha \pi}{\alpha \pi} \right) = 0, \tag{23} \]

and from (16)-(19) consumption allocations are

\[c_{11}^t = n_1 \frac{\mu - 1 + \alpha \pi}{(\mu - 1) [1 - (1 - \alpha)\pi] + \alpha \pi}, \tag{24} \]

\[c_{12}^t = n_2 \frac{\mu - 1 + \alpha \pi}{\alpha \pi}, \tag{25} \]

\[c_{22}^t = n_2 \frac{1}{\mu}, \tag{26} \]

\[c_{21}^t = n_1 \frac{\alpha \pi}{(\mu - 1) [1 - (1 - \alpha)\pi] + \alpha \pi}. \tag{27} \]

From (22)-(27), if the money supply is fixed for all $t$ ($\mu = 1$), which implies that $\delta = 1$ from (21), so that the distribution of money balances across the population is
uniform, then \( n_1 = n_2 = n^* \), where \( n^* \) is determined by

\[
-v'(n^*) + \beta u'(n^*) = 0, \tag{28}
\]

and consumption is \( n^* \) for all agents in each period. However if \( \mu \neq 1 \) then consumption will be different for consumers who purchase goods at a particular location, depending on their home location, and consumption will also differ for consumers from a given location depending on where they purchase goods. Thus, if the money supply is not constant, then agents face uninsured consumption risk. A higher money growth rate implies, from (24)-(27), that consumers from households located on connected islands consume larger shares of output, and consumers from unconnected islands consume smaller shares. As \( \mu \to \infty \), connected-island households consume all output.

We can derive the effect of a change in the money growth factor \( \mu \) on \( n_1 \) and \( n_2 \), at least for \( \mu = 1 \). That is, totally differentiating (22) and (23) and evaluating derivatives at \( \mu = 1 \), we obtain

\[
\frac{dn_1}{d\mu} = \frac{\beta[-v''(n^*) + \beta u''(n^*)] \left\{ -\frac{(1-\alpha)(2-\pi)[n^* u''(n^*) + u'(n^*)]}{\alpha} + \beta u'(n^*) \right\}}{\nabla}, \tag{29}
\]

\[
\frac{dn_2}{d\mu} = \left\{ \frac{\beta(2 - \pi)[-v''(n^*) + \beta u''(n^*)] [n^* u''(n^*) + u'(n^*)]}{-\beta^2 \alpha \pi \left[ u''(n^*) + \frac{u'(n^*)}{n^*} \right] u'(n^*)} \right\}, \tag{30}
\]

where

\[
\nabla = [-v''(n^*) + \beta u''(n^*)] \left\{ \beta(1 - \pi) u''(n^*) - \beta \pi \frac{u'(n^*)}{n^*} - v''(n^*) \right\} > 0.
\]

In general, we cannot sign \( \frac{dn_1}{d\mu} \) and \( \frac{dn_2}{d\mu} \), though \( \frac{dn_2}{d\mu} < 0 \) if the coefficient of relative risk aversion is less than 1 (the substitution effect of an increase in the effective real wage dominates the income effect). A key result is that an increase in the money
growth rate when $\mu = 1$ will reduce aggregate labor supply and output. That is, from (29) and (30) we obtain

$$
\alpha \frac{dn_1}{d\mu} + (1 - \alpha) \frac{dn_1}{d\mu} = \frac{\beta \alpha u'(n^*) \left\{ -\n'(n^*) + \beta[1 - (1 - \alpha)\pi]u'(n^*) - \frac{\beta(1 - \alpha)\pi u'(n^*)}{n^*} \right\}}{\nabla} < 0.
$$

(31)

We get this result for the standard reason - inflation is a tax on labor supply, and thus higher inflation tends to reduce employment and output.

**Optimal Money Growth**

There are two distortions that arise here. The first is a standard monetary distortion; that is, with a fixed money supply and discounting, the rate of return on money tends to be too low and less labor is supplied than is optimal. Second, consumption will differ among agents if $\mu \neq 1$. For example, if $\mu > 1$ then the price of consumption goods will tend to be higher on connected than on unconnected islands, so that consumers who purchase goods on connected islands will tend to consume less than those who purchase on unconnected islands. Further, if $\mu > 1$ then consumers from connected islands will have more money than will consumers from unconnected islands, and so the former set of consumers will be able to purchase more goods at a given location. The reverse is true if $\mu < 1$. While the first distortion will induce a benefit from deflation, the second distortion will induce costs of a non-constant money supply. As we will show in what follows, this implies that a small amount of deflation is optimal, but deflation at the rate of time preference ($\mu = \beta$) is either infeasible or suboptimal. These results are similar in flavor to what holds in sticky price models, but of course prices are perfectly flexible here; the key frictions are that monetary policy has distributional effects and goods cannot be moved across locations.

Now, suppose that we look for an optimal monetary growth rule in the class of constant growth polices with constant $\delta$ as in (21). We assume binding cash-in-
advance constraints, so that an equilibrium solution is characterized by (22) and (23). Note that all cash-in-advance constraints will bind in the neighborhood of \( \mu = 1 \). If we weight expected utilities of households equally, then the optimal money growth rate is the solution to the problem \( \max_\mu W(\mu) \) subject to (22)-(27), where

\[
W(\mu) = \begin{cases}
\alpha [1 - (1 - \alpha) \pi] u(c_{11}^1) + \alpha (1 - \alpha) \pi u(c_{12}^1) \\
+ (1 - \alpha)(1 - \alpha \pi) u(c_{22}^2) + (1 - \alpha) \alpha \pi u(c_{21}^2) \\
- \alpha v(n_1) - (1 - \alpha) v(n_2)
\end{cases}
\]

As is standard, an equilibrium requires \( \mu \geq \beta \). A non-standard restriction, which arises from the requirement that consumption be nonnegative for all agents is, from (24)-(27),

\( \mu > \frac{1 - \pi}{1 - (1 - \alpha) \pi} \).

This constraint implies that there are circumstances under which a Friedman rule is not feasible. That is, if

\( \beta \leq \frac{1 - \pi}{1 - (1 - \alpha) \pi} \),

then an equilibrium does not exist when \( \mu = \beta \). The possibility of an infeasible Friedman rule arises because, if \( \alpha \) is sufficiently small, then the taxes required to support a Friedman rule deflation would be greater than the money balances that households on connected islands have available at the beginning of the period.

Now, if we differentiate (32) using (24)-(27) and evaluate the derivative for \( \mu = 1 \), we obtain

\[
W'(1) = A + (1 - \beta) u'(n^*) \left[ \alpha \frac{dn_1}{d\mu} + (1 - \alpha) \frac{dn_1}{d\mu} \right],
\]

where \( A \) is the effect of a change in \( \mu \) on welfare caused by the increase in consumption risk arising from the redistribution of consumption goods among agents. The remaining portion of the change in welfare is the net effect on welfare of the change in labor supply resulting from a change in \( \mu \). It is straightforward to show that \( A = 0 \), that is since consumption is equal across agents when \( \mu = 1 \), the first-order effect of
a change in $\mu$ on consumption risk is nil. Therefore, the net effect on welfare when $\mu = 1$ is determined by the effect on aggregate output, which from (31) is negative. Therefore, a small reduction in the money growth rate from zero will increase welfare.

**Numerical Exercises**

Solutions can be computed by using (22) and (23) to solve for $n_1$ and $n_2$ given $\mu$, and then we can solve for consumptions and welfare from (24)-(27) and (32). For now, we use some more-or-less arbitrary parameter values to obtain a feel for the quantitative results the model can deliver.

We let $u(c) = \frac{c^{\gamma} - 1}{1-\gamma}$, with $\gamma > 0$ and $v(n) = n$. To begin, let $\beta = .99$, and $\alpha = \pi = .5$. Figures 1-3 show results for different levels of risk aversion, since curvature in the utility function will be critical to determining the effects of money growth, which has important effects on consumption risk. In Figure 1, we graph welfare relative to optimal money growth, measured in units of consumption relative to what is achieved with an optimal money growth rate, for different levels of the coefficient of relative risk aversion. Note that the optimal money growth rate increases with the coefficient of relative risk aversion, and that for a fairly moderate level of risk aversion ($CRRA = 2$) a fixed money supply is very close to optimal. That is, it does not take a high degree of risk aversion for consumption risk to become the dominant force in determining optimal monetary policy. Higher risk aversion of course also increases the welfare costs of deviations from the optimal money growth rate. In Figure 2 we show the same picture as in Figure 1, but we include higher levels for the money growth rate. With $CRRA = .5$, the welfare loss from a 10% per period inflation is somewhat more than 1% of consumption, but this number increases to more than 7% of consumption for $CRRA = 3$, a cost which is very large relative to what is typically obtained in the literature. The welfare cost of inflation also depends critically on $\alpha$ and $\pi$. The larger is $\alpha$, the lower is the cost of inflation. Note in particular that
\( \alpha = 1 \) gives us a standard cash-in-advance model. Since \( \pi \) determines the speed of diffusion of a money injection through the economy, higher \( \pi \) will imply a lower cost of inflation. Figure 3 shows how money growth affects consumption risk. Note, for example, that for a 1% money growth rate, there is a very large difference between the consumption of agents who move from a connected to unconnected island \( (c_{12}) \), and those who move from an unconnected to a connected island \( (c_{21}) \).

4. STOCHASTIC MONEY GROWTH

The previous section tells us something about optimal monetary policy, in the class of policies that yield the same consumption and employment allocation in each period. We have also learned something about the costs of suboptimal money growth rates. The purpose of this section is to study the effects of suboptimal random monetary policy. We compute equilibria for cases where the money growth rate is random, and study the impulse responses to money growth shocks.

Assume that the gross money growth rate \( \mu \) follows a first-order Markov process, and as before let \( \delta \) denote the ratio of per capita money balances on connected islands.
Figure 2: Welfare

Figure 3: Consumption

Fig. 2.

Fig. 3.
to per capita money balances on unconnected islands. Then, the state is described by \((\mu, \delta)\), and assuming that cash-in-advance constraints hold with equality, the law of motion for \(\delta\) is, using (10), (11), and (20),

\[
\delta' = \frac{\alpha[\mu' - (1 - \alpha)\pi]\delta + (1 - \alpha)(\mu' - 1 + \alpha\pi)}{\alpha^2\pi\delta + \alpha(1 - \alpha\pi)},
\]

where primes denote variables dated \(t + 1\). Therefore, if consumers spend all of the household’s cash balances each period, then the stochastic process for \(\delta\) is exogenous, which makes computing equilibria relatively straightforward. It is clear from (33) that there is persistence in \(\delta\), due to the fact that it takes time for a money shock to diffuse. The diffusion rate is governed by \(\pi\). Note that, if \(\pi = 0\), then (33) gives

\[
\delta' = \frac{\mu' - 1 + \alpha}{\alpha},
\]

in which case the distribution of money balances across the population is determined only by the current money growth rate, and there is no persistence.

In this environment, the first-order conditions from the optimization problems of households on connected and unconnected islands, respectively, yield

\[
-v'(n_{11}) + \beta \int \left\{ \frac{p_{11}[1 - (1 - \alpha)\pi]u'(c_{1t+1})}{p_{1,t+1}} + \frac{p_{11}(1 - \alpha)\pi u'(c_{1t+1})}{p_{2,t+1}} \right\} dF(\mu'; \mu) = 0,
\]

\[
-v'(n_{21}) + \beta \int \left\{ \frac{p_{21}(1 - \alpha\pi)u'(c_{21})}{p_{1,t+1}} + \frac{p_{21}(1 - \alpha\pi)u'(c_{21})}{p_{2,t+1}} \right\} dF(\mu'; \mu) = 0,
\]

where \(F(\mu'; \mu)\) is the distribution of \(\mu'\) conditional on \(\mu\). Let \(n_i(\delta, \mu)\) denote the level of employment in state \((\delta, \mu)\), where \(i = 1\) denotes a connected island and \(i = 2\) denotes an unconnected island. Then, substituting in (34) and (35) using the equilibrium conditions (12) and (13) and using (20), we get
\[
\begin{align*}
&+\beta \int \left\{ -v'[n_1(\delta, \mu)]n_1(\delta, \mu) \\
&\quad + (1 - \alpha)\pi u' \left[ n_2(\delta', \mu'[\delta']) \right] \cdot \frac{[n_1(\delta', \mu')][1 - (1 - \alpha)\pi][\delta + (1 - \alpha)\pi]}{\phi_1(\mu', \delta)} \right\} dF(\mu'; \mu) \\
&= 0, \\
&+\beta \int \left\{ \alpha \pi u' \left[ n_1(\delta, \mu) \right] \cdot \frac{[n_1(\delta', \mu')][\alpha \pi \delta + 1 - \alpha \pi]}{\phi_1(\mu', \delta)} \right\} dF(\mu'; \mu) = 0, \\
\end{align*}
\]

and the consumption allocations are

\[
\begin{align*}
c^{11}(\delta, \mu) &= \frac{n_1(\delta, \mu)}{[1 - (1 - \alpha)\pi][\delta + (1 - \alpha)\pi]}, \\
c^{12}(\delta, \mu) &= \frac{n_2(\delta, \mu)}{\alpha \pi \delta + 1 - \alpha \pi}, \\
c^{22}(\delta, \mu) &= \frac{n_2(\delta, \mu)}{\alpha \pi \delta + 1 - \alpha \pi}, \\
c^{21}(\delta, \mu) &= \frac{n_1(\delta, \mu)}{[1 - (1 - \alpha)\pi][\delta + (1 - \alpha)\pi]}. \\
\end{align*}
\]

In (36) and (37) \( \delta' \) is determined by (33) and the functions \( \phi_i(\mu', \delta) \) for \( i = 1, 2 \), are defined by

\[
\begin{align*}
\phi_1(\mu', \delta) &\equiv \{ \mu' [1 - (1 - \alpha)\pi] - (1 - \alpha)\pi (1 - \pi) \} \delta \\
&\quad + (1 - \alpha) \left\{ (\mu' - 1) \frac{[1 - (1 - \alpha)\pi]}{\alpha} + \pi (2 - \pi) \right\}, \\
\phi_2(\mu', \delta) &\equiv \alpha \pi (\mu + 1 - \pi) \delta + (1 - \alpha \pi) (1 - \pi) + \mu' \pi (1 - \alpha). \\
\end{align*}
\]
Numerical Exercises

As with the constant money growth experiments, we use \( u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma} \), with \( \gamma > 0 \) and \( v(n) = n \). We set \( \beta = .99, \gamma = 1.5, \alpha = .2, \) and \( \pi = .1 \). Thus, we use parameter values such that monetary shocks will initially affect few agents and the shocks will have highly-persistent effects. As well, we will look only at examples where money growth shocks are i.i.d., as in this case there would be no effect of a money shock on employment, output, and consumption in the special case where there are no distribution effects (\( \alpha = 1 \)). In this case, all of the persistence in the effects of money shocks will come from persistent effects on the distribution of money balances across the population.

In the first experiment, we look at an economy where the money growth factor takes on values over a uniform grid on \([1, 1.02]\), with equal probability mass on each point in the grid. Therefore, the mean gross growth rate of the money stock is 1.01. Since the money growth factor is i.i.d., the state variable is \( \delta \), and Figures 4 and 5 show the optimal decision rules for employment and consumption, respectively. We then study an experiment where we suppose that the money growth factor has been at 1.01 indefinitely, and then study the impulse responses to a shock where the money growth factor increases for one period to 1.02 and then returns to 1.01 forever. Figures 6 and 7 show the impulse responses of employment and consumption. Note that the sectoral effects are large in that employment in connected locations increases substantially and employment in unconnected locations falls substantially, and there is a significant increase in the dispersion of consumption across agents. However, there is only a very small increase in aggregate output and employment. Therefore, small money shocks can be important for sectoral consumption and employment (and presumably welfare), but have little effect on aggregates.

For the second experiment, parameters are identical to what we used for the first
experiment, but now we increase the mean and variance of the money growth rate. In this case, the grid for the money growth factor is on \([1, 2]\), again with equal probability mass on each point in the grid. Now, the experiment is to hold the money growth factor at \(1.5\) indefinitely, increase the money growth factor to \(2\) for one period, and then look at the impulse responses. In Figures 8 and 9 we show impulse responses for employment and consumption, respectively. Note in Figure 8 that the response of aggregate employment (equal to output here) is proportionately much larger than was the case for a small money shock in the previous experiment. As before, there are large effects on sectoral employment (Figure 8) and on the dispersion in consumption across agents (in Figure 9). Thus, large money shocks can matter proportionately more for aggregates than do small money shocks.

**Fig. 4.**
Figure 5: Decision Rules, Small Money Shocks

Figure 6: Impulse Responses, 1% money growth shock

Fig. 5.

Fig. 6.

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Figure 7: Impulse Responses, 1% money growth shock

Figure 8: Impulse Responses, 50% to 100% Money Growth Shock

Fig. 7.

Fig. 8.
CONCLUSION

We have constructed a model of heterogeneous households that captures some novel distributional effects of monetary policy. In the model, some households find themselves on the receiving end of a money injection by the central bank, and some do not. In general there will be price dispersion across markets generated by monetary policy, and as a result monetary policy can produce uninsured consumption risk. This consumption risk is important in determining optimal money growth rates and affects the response of the economy to aggregate money shocks.

We showed that, for moderate levels of risk aversion, a constant money stock could be very close to optimal, and the welfare cost of a small inflation could be very large. Small money growth shocks can have large effects on sectoral employment and the dispersion of consumption, but tend to have small effects on aggregates. However,
large money growth shocks can have proportionately large effects on aggregate output, employment, and consumption.

REFERENCES


