Variance Aversion in the Small and the Large

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Abstract

This paper proves an analogue of Pratt's theorem for two measures of risk aversion for mean-variance preferences. A direct link is established between these measures and standard measures of risk aversion. Implications for problems of choice under uncertainty such as portfolio choice problems are derived.

Keywords: Risk aversion, Mean-variance preferences, Portfolio choice

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1. Introduction

This paper establishes the equivalence of two measures of risk aversion for a general class of mean-variance preferences. It further derives implications for decision making under uncertainty, and establishes portfolio characterization of risk aversion for mean-variance preferences, which allows for comparison of agents' attitudes toward risk based on the choices they make under uncertainty. The motivation for studying mean-variance preferences is twofold: 1) they are widely used in finance; and 2) Epstein (1985) has shown that in the class of Machina's (1982) non-expected utility preferences only mean-variance preferences satisfy appropriate decreasing-absolute-risk-aversion conditions.

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I introduce two measures of variance aversion, and analyze their connection to existing measures of risk aversion. An analogue of Pratt's (1964) theorem is proven for the mean-variance preferences. And finally, equivalence is established between comparing these measures and comparing choices that agents make under uncertainty.

The first measure, coefficient of variance aversion, is twice the slope of the agent's mean-variance indifference curve. It is closely related to the Arrow-Pratt coefficient of risk aversion, and coincides under certain conditions with Epstein's (1985) measure of risk aversion for a general class of non-expected utility preferences in Machina's (1982) framework. The second measure, the partial variance compensation, is the mean-variance counterpart of partial risk compensation introduced by Ross (1982) for expected utility preferences; and thus links the result of this paper to the stronger notion of risk aversion of Ross.

The main theorem establishes an equivalence of the introduced measures of variance aversion. I also show that "agent 1 is (weakly) more variance averse than agent 2 if and only if agent 1 always chooses a prospect (portfolio) with (weakly) less variance than in the one chosen by agent 2".

I describe the general framework and define the measures of variance aversion in section 2. Section 3 is the main theorem. In section 4, I analyze the relationship of the introduced measures to existing measures of risk aversion. Section 5 is dedicated to implications for choice under uncertainty, and in particular, portfolio choice problem.

2. Assumptions and Definitions

Mean-variance preferences are represented by the utility function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$; so that $u(E, \sigma^2)$ is the utility of a random prospect with mean E and variance σ^2. I assume that

(A1) u is twice continuously differentiable;

(A2) $\frac{\partial u}{\partial E} > 0;$ $\frac{\partial u}{\partial \sigma^2} \leq 0.$

Consider two agents 1 and 2 with mean-variance utility functions u_1 and u_2, respectively.

Definition 1 *Agent 1 is (weakly) more variance averse than agent 2 if*

$$u_2(E,\sigma^2) \geq u_2(E_0,\sigma_0^2) \text{ with } E_0 \geq E \quad \Rightarrow \quad u_1(E,\sigma^2) \geq u_1(E_0,\sigma_0^2).$$

That is, agent 1 is (weakly) more variance averse than agent 2 if 1 rejects a prospect with higher variance (and higher mean) whenever 2 rejects it.

Now introduce the following two measures of variance aversion:

Definition 2 *Coefficient of Variance Aversion is* $\quad r(E,\sigma^2) = -\,2\left.\frac{\frac{\partial u}{\partial \sigma^2}}{\frac{\partial u}{\partial E}}\right|_{(E,\sigma^2)}.$

This is twice the marginal rate of substitution between E and σ^2. That is twice the slope of the agent's indifference curve in the mean-variance space with variance on the horizontal axes.

Definition 3 *Partial Variance Compensation is* $\chi(E,\sigma^2;\Delta) \in \mathbb{R}_+$ *s.t.*

$$u(E,\sigma^2) = u(E-\chi(E,\sigma^2;\Delta),\sigma^2-\Delta).$$

That is how much mean the agent is willing to give up to reduce the variance by Δ.

3. Main Theorem

If (A1) holds, then for small Δ, partial variance compensation χ is proportional to Δ with coefficient of proportionality equal to $\frac{r}{2}$. To formalize this, I use the following notation: $f(\delta) \cong g(\delta)$ for small δ, if $\lim\limits_{\delta\to 0}\frac{f(\delta)-g(\delta)}{\delta} = 0$.

Lemma 4 *If (A1) holds, then for small* Δ, $\chi(E, \sigma^2; \Delta) \cong \frac{1}{2} r(E, \sigma^2) \cdot \Delta$.

Proof. Follows from first-order Taylor's expansion of $u(E - \chi(E, \sigma^2; \Delta), \sigma^2 - \Delta)$. ■

We are now ready to prove the main theorem, which establishes that the introduced measures capture our notion of variance aversion and are equivalent.

Theorem 5 *If (A1), (A2) are satisfied, the following are equivalent:*

1. $r_1(E, \sigma^2) \geq r_2(E, \sigma^2) \quad \forall E, \sigma^2$;

2. $\chi_1(E, \sigma^2; \Delta) \geq \chi_2(E, \sigma^2; \Delta) \quad \forall E, \sigma^2; \quad \forall \Delta \in [0, \sigma^2]$;

3. *Agent 1 is (weakly) more variance averse than agent 2.*

Proof. See Appendix ■

4. Relation to Existing Measures of Risk Aversion

4.1. Expected Utility Framework

Whenever mean and variance are sufficient to characterize expected utility preferences over risky prospects (for example, when one restricts attention only to quadratic von Neumann-Morgenstern utility functions or only to normally distributed random variables), variance fully characterizes risk, and partial variance compensation coincides with partial risk compensation, introduced by Ross (1982).

Suppose that mean-variance utility function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and von Neumann-Morgenstern utility function $v : \mathbb{R} \to \mathbb{R}$ represent the same preferences. That is, $Ev(\widetilde{x}) \geq Ev(\widetilde{y})$ if and only if $u(E(\widetilde{x}), Var(\widetilde{x})) \geq u(E(\widetilde{y}), Var(\widetilde{y}))$.

Ross (1982) defines partial risk compensation $\pi(\widetilde{w}, \widetilde{\varepsilon})$ by

$$Ev(\widetilde{w} + \widetilde{\varepsilon}) = Ev(\widetilde{w} - \pi(\widetilde{w}, \widetilde{\varepsilon})) \tag{1}$$

where $\widetilde{w}$ and $\widetilde{\varepsilon}$ are random variables, and $E(\widetilde{\varepsilon}|\widetilde{w}) = 0$. Note that (1) is equivalent to

$$u(E\widetilde{w}, Var(\widetilde{w}) + Var(\widetilde{\varepsilon})) = u(E\widetilde{w} - \pi(\widetilde{w}, \widetilde{\varepsilon}), Var(\widetilde{w})) \tag{2}$$

And by definition, $\pi(\widetilde{w}, \widetilde{\varepsilon}) = \chi(E\widetilde{w}, Var(\widetilde{w}) + Var(\widetilde{\varepsilon}); Var(\widetilde{\varepsilon}))$.

Thus whenever preferences over a given class of distributions have both an expected utility representation and a mean-variance representation,

2. $\chi_1(E, \sigma^2; \Delta) \geq \chi_2(E, \sigma^2; \Delta) \qquad \forall E, \sigma^2; \qquad \forall \Delta \in [0, \sigma^2]$

implies

2*. $\pi_1(\widetilde{w}, \widetilde{\varepsilon}) \geq \pi_2(\widetilde{w}, \widetilde{\varepsilon}) \qquad \forall \widetilde{w}, \widetilde{\varepsilon}$ such that $E(\widetilde{\varepsilon}|\widetilde{w}) = 0$.

Moreover, if for any $(E, \sigma^2; \Delta)$ one can pick $\widetilde{w}$ with $E\widetilde{w} = E$, $Var(\widetilde{w}) = \sigma^2$, and $\widetilde{\varepsilon}$ with $E(\widetilde{\varepsilon}|\widetilde{w}) = 0$, $Var(\widetilde{\varepsilon}) = \Delta$, then (**2.**) is *equivalent* to (**2*.**).

Thus, the main theorem provides a near equivalence relationship of the introduced measures of variance aversion to Ross' (1982) stronger measures of risk aversion over the intersections of the classes of expected utility and mean-variance preferences. Note however that Ross' analogue of Pratt's theorem may fail in a restricted class of distributions. In particular, the theorem fails if one restricts attention to normally distributed random variables.

Ross' (1982) results also imply that whenever mean and variance are sufficient to characterize expected utility preferences, if 1 is more variance averse than 2, then 1 is more risk averse (in Arrow-Pratt sense) than 2. In particular, for the class of environments where preferences can be represented by expected quadratic utility $E\left[-\left(\widetilde{x}-a\right)^2\right]$, the coefficient of variance aversion is $r=\frac{1}{a-Ex}$, while the Arrow-Pratt coefficient of risk aversion is $r^a=\frac{1}{a-x}$.

Clearly, in that case, $\quad r_1 \geq r_2 \;\; \forall\widetilde{x} \quad \Longleftrightarrow \quad r_1^a \geq r_2^a \;\; \forall x.$

4.2. Non-Expected Utility Framework

Epstein (1985) introduces the following measure of risk aversion for Machina's (1982) general class of non-expected utility preferences:

$$A\left(y,F\right)=\frac{-U_{11}\left(y,F\right)}{\int U_1\left(w,F\right)\;dF\left(w\right)}, \tag{3}$$

where U is local utility function, and U_1 is partial derivative of $U\left(y,F\right)$ with respect to y. Epstein shows that under decreasing-absolute-risk-aversion conditions (non-expected) utility functional V can be expressed in the form $V\left(F\right)=u\left(\mu\left(F\right),\sigma^2\left(F\right)\right)$, and that

$$A\left(y,F\right)=-2\frac{\frac{\partial u}{\partial\sigma^2}\left(\mu\left(F\right),\sigma^2\left(F\right)\right)}{\frac{\partial u}{\partial E}\left(\mu\left(F\right),\sigma^2\left(F\right)\right)}, \tag{4}$$

which is exactly the coefficient of variance aversion defined above. Thus, our main theorem provides a new link between measures of risk aversion in the standard expected utility framework and in Machina's (1982) non-expected utility framework.

See Kusuda (2001) for further analysis of relations between various measures of risk aversion.

5. Implications for Choice under Uncertainty

In this section, I follow the approach of Wang and Werner (1994).

Theorem 5 has direct implication to problems of choice under uncertainty with mean-variance preferences (CAPM, etc.), because

agent 1 is (weakly) more risk averse than agent 2 if and only if

agent 1 always chooses prospect with no more variance than in the one chosen by agent 2 in the same situation.

To formalize the above statement I need to introduce the following relationship between two sets of payoffs, C_1 and C_2:

Definition 6 $C_1 \leqslant_{var} C_2$ *if*

$$h_1 \in C_1,\ h_2 \in C_2,\ Var(h_2) \leq Var(h_1) \qquad \Rightarrow \qquad h_2 \in C_1, h_1 \in C_2.$$

Now consider a general problem of choice under uncertainty:

$$\max_{h \in CS} u_i \left(Eh, Var(h)\right), \tag{5}$$

where CS is an arbitrary constraint set; and denote the set of solutions by $C_i := \arg\max_{h \in CS} u_i \left(Eh, Var(h)\right)$.

Theorem 7 *If (A1), (A2) are satisfied, the following are equivalent:*

3. *Agent 1 is (weakly) more variance averse than agent 2;*

4. $C_1 \leqslant_{var} C_2$, *where* $C_i = \arg\max_{h \in CS} u_i \left(Eh, Var(h)\right)$, *for any* CS.

Proof. See Appendix ■

5.1. Implications for Portfolio Choice

Everything I have established so far in this section can also be applied to a less general, but more economically interesting class of problems - portfolio choice problems.

Definition 8 ***Portfolio choice problem*** *(PCP) with short-sales constraints and 2 securities with random payoffs* h_1 *and* h_2 *is:*

$$\max_{\alpha} u_i\left(Eh, Var(h)\right) \qquad s.t. \qquad h = \alpha h_1 + (1-a)\, h_2, \qquad \alpha \in [0,1]$$

Denote the set of solutions by C_i.

Theorem 9 *If (A1), (A2) are satisfied, the following are equivalent:*

3. *Agent 1 is (weakly) more variance averse than agent 2;*

4*. $C_1 \leqslant_{var} C_2$ *(where* C_i *is solution of PCP)* *for any PCP.*

Proof. See Appendix ■

This theorem not only demonstrates that our measures of variance aversion have direct implications for portfolio choice problems, but also allows portfolio characterization of variance aversion (see Wang and Werner (1994)).

6. Appendix: Proofs

Proof. Theorem 5.

2 ⇒ **1**. The proof is by contradiction, using Lemma 4.

Suppose $\chi_1(E, \sigma^2; \Delta) \geq \chi_2(E, \sigma^2; \Delta)$ for $\forall E, \sigma^2$; $\forall \Delta \in [0, \sigma^2]$, but $\exists\ \bar{E}, \bar{\sigma}^2$ such that $r_1(\bar{E}, \bar{\sigma}^2) < r_2(\bar{E}, \bar{\sigma}^2)$.

Take R such that $r_1(\bar{E}, \bar{\sigma}^2) < R < r_2(\bar{E}, \bar{\sigma}^2)$. Then by the lemma, $\exists\ \delta > 0$, such that $\chi_1(\bar{E}, \bar{\sigma}^2; \Delta) < \frac{1}{2} R \cdot \Delta$ and $\chi_2(\bar{E}, \bar{\sigma}^2; \Delta) > \frac{1}{2} R \cdot \Delta$ for any $\Delta < \delta$. I.e., $\chi_1(\bar{E}, \bar{\sigma}^2; \Delta) < \chi_2(\bar{E}, \bar{\sigma}^2; \Delta)$, and we have reached a contradiction.

3 ⇒ **2**. The proof is by contradiction.

Suppose (**3**) holds, but $\exists\ E_0, \sigma_0^2, \Delta$ s.t. $\chi_1(E_0, \sigma_0^2; \Delta) < \chi_2(E_0, \sigma_0^2; \Delta)$.

Define $\widehat{E} := E_0 - \frac{\chi_1(E_0,\sigma_0^2;\Delta)+\chi_2(E_0,\sigma_0^2;\Delta)}{2}$, and $\widehat{\sigma}^2 := \sigma_0^2 - \Delta$.

Note that $\widehat{E} \in (E_0 - \chi_1, E_0 - \chi_2)$ and $E_0 > \widehat{E}$.

$u_2(\widehat{E}, \widehat{\sigma}^2) > u_2(E_0 - \chi_2, \widehat{\sigma}^2) = u_2(E_0 - \chi_2(E_0, \sigma_0^2; \Delta), \sigma_0^2 - \Delta) = u_2(E_0, \sigma_0^2)$, but

$u_1(\widehat{E}, \widehat{\sigma}^2) < u_1(E_0 - \chi_1, \widehat{\sigma}^2) = u_1(E_0 - \chi_1(E_0, \sigma_0^2; \Delta), \sigma_0^2 - \Delta) = u_1(E_0, \sigma_0^2)$, which is a contradiction to (**3**).

1 $\Rightarrow$ **3**. Differentiate agent 1's utility along agent 2's indifference curve.

Consider arbitrary $E, \sigma^2, E_0, \sigma_0^2$, such that $E_0 \geq E$ and $u_2(E, \sigma^2) \geq u_2(E_0, \sigma_0^2)$. Then, since u_2 is continuous, $\exists\ \widehat{\sigma}^2 \leq \sigma_0^2$, such that $u_2(E, \sigma^2) = u_2(E_0, \widehat{\sigma}^2)$. Also by continuity of u_2, we can parameterize the indifference curve of agent 2 going through (E, σ^2) (and $(E_0, \widehat{\sigma}^2)$) as $(m, v) = (\phi(v), v)$, where ϕ is the implicit function that solves $u_2(x, v) = u_2(E, \sigma^2)$ for x. Note that ϕ is a weakly increasing function. By implicit function theorem, ϕ is differentiable.

By construction, $\frac{du_2(\phi(v),v)}{dv} = \frac{\partial u_2(\phi(v),v)}{\partial E}\phi'(v) + \frac{\partial u_2(\phi(v),v)}{\partial \sigma^2} = 0$.

By assumption of (1), $-\frac{\frac{\partial u_1}{\partial \sigma^2}}{\frac{\partial u_1}{\partial E}} \geq -\frac{\frac{\partial u_2}{\partial \sigma^2}}{\frac{\partial u_2}{\partial E}}$. Hence, $-\frac{\partial u_1}{\partial \sigma^2}\frac{\partial u_2}{\partial E}\phi' \geq -\frac{\partial u_2}{\partial \sigma^2}\frac{\partial u_1}{\partial E}\phi'$. But we have $\frac{\partial u_2}{\partial E}\phi' = -\frac{\partial u_2}{\partial \sigma^2}$. So, $\frac{\partial u_1}{\partial \sigma^2}\frac{\partial u_2}{\partial \sigma^2} \geq -\frac{\partial u_2}{\partial \sigma^2}\frac{\partial u_1}{\partial E}\phi'$.

Consider the following two cases:

a. $\frac{\partial u_2}{\partial \sigma^2} = 0$. Then 2's indifference curve is locally horizontal, $\phi' = 0$, and since $\frac{\partial u_1}{\partial \sigma^2} \leq 0$, $\quad \frac{du_1(\phi(v),v)}{dv} = \frac{\partial u_1(\phi(v),v)}{\partial E}\phi'(v) + \frac{\partial u_1(\phi(v),v)}{\partial \sigma^2} \leq 0$.

b. $\frac{\partial u_2}{\partial \sigma^2} > 0$. Then $-\frac{\partial u_1}{\partial \sigma^2} \geq \frac{\partial u_1}{\partial E}\phi'$, and $\frac{du_1(\phi(v),v)}{dv} = \frac{\partial u_1(\phi(v),v)}{\partial E}\phi'(v) + \frac{\partial u_1(\phi(v),v)}{\partial \sigma^2} \leq 0$.

This implies $u_1(E, \sigma^2) \geq u_1(E_0, \widehat{\sigma}^2) \geq u_1(E_0, \sigma_0^2)$. ■

Proof. Theorem 7.

3 $\Rightarrow$ **4**. Take any $h_1 \in C_1$, $h_2 \in C_2$, such that $Var\,(h_1) \geq Var\,(h_2)$. Then $E\,(h_1) \geq E\,(h_2)$ (otherwise $h_1 \notin C_1$). Since $h_2 \in C_2$, $u_2(h_2) \geq u_2(h_1)$; from (**3**) $u_1(h_2) \geq u_1(h_1)$;

and $h_2 \in C_1$.

To show that $h_1 \in C_2$, suppose not, i.e. $u_2(h_2) > u_2(h_1)$. Take (E,σ^2) such that $E = E(h_1) + \varepsilon$, $\sigma^2 = Var(h_1)$, with $\varepsilon > 0$ small enough such that $u_2(h_2) > u_2(E,\sigma^2) > u_2(h_1)$. But then $u_1(h_2) \geq u_1(E,\sigma^2) > u_1(h_1)$, and $h_1 \notin C_1$, which is a contradiction.

4 $\Rightarrow$ **3**. Take any (E,σ^2), (E_0,σ_0^2) with $E_0 \geq E$ such that $u_2(E,\sigma^2) \geq u_2(E_0,\sigma_0^2)$, and consider $CS = \{h, h_0\}$ with $E(h) = E$, $Var(h) = \sigma^2$, $E(h_0) = E_0$, $Var(h_0) = \sigma_0^2$. Then $h \in C_2$, and since $Var(h_0) \geq Var(h)$, $h \in C_1$. That is $u_1(E,\sigma^2) \geq u_1(E_0,\sigma_0^2)$.

■

Proof. Theorem 9.

3 $\Rightarrow$ **4***. This follows immediately from the previous theorem.

4$^*\Rightarrow$ **3**. Suppose not. Then $\exists$ (E,σ^2) such that $r_2(E,\sigma^2) > r_1(E,\sigma^2)$. Take R such that $r_1(E,\sigma^2) < R < r_2(E,\sigma^2)$. By (A1), $\exists\, \varepsilon > 0$ s.t. $r_1(\bar{E},\bar{\sigma}^2) < R < r_2(\bar{E},\bar{\sigma}^2)$ for $\forall(\bar{E},\bar{\sigma}^2)$ s.t. $\left\|(\bar{E},\bar{\sigma}^2) - (E,\sigma^2)\right\| < \varepsilon$.

Take $\delta \in \left(0, \frac{\varepsilon}{2}\right)$ and define $\Delta = R\sqrt{\sigma^2+\delta}\left(\sqrt{\sigma^2+\delta} - \sigma\right)$. Take δ small enough so that $\Delta < \frac{\varepsilon}{2}$, and construct two perfectly positively correlated random variables h_1 and h_2 such that $E(h_1) = E$, $Var(h_1) = \sigma^2$, $E(h_2) = E + \Delta$, $Var(h_2) = \sigma^2 + \delta$, $\rho(h_1,h_2) = 1$.

Construct random variable $h_\alpha = \alpha h_1 + (1-a) h_2$. Then $E(h_\alpha) = E + (1-\alpha)\Delta$, and $Var(h_\alpha) = \alpha^2\sigma^2 + (1-\alpha)^2(\sigma^2+\delta) + 2\alpha(1-\alpha)\sigma\sqrt{\sigma^2+\delta}$. Since $\frac{dE(h_\alpha)}{d\alpha} = -\Delta$ and $\frac{dVar(h_\alpha)}{d\alpha} = 2\left[\alpha\left(\sqrt{\sigma^2+\delta}-\sigma\right)^2 - \sqrt{\sigma^2+\delta}\left(\sqrt{\sigma^2+\delta}-\sigma\right)\right]$, we have:

$$\frac{\partial u_i(E(h_\alpha),Var(h_\alpha))}{\partial\alpha} =$$

$$= -\Delta\frac{\partial u_i(h_\alpha)}{\partial E} + 2\left(\alpha\left(\sqrt{\sigma^2+\delta}-\sigma\right)^2 - \sqrt{\sigma^2+\delta}\left(\sqrt{\sigma^2+\delta}-\sigma\right)\right)\frac{\partial u_i(h_\alpha)}{\partial\sigma^2}$$

$$= -\sqrt{\sigma^2+\delta}\left(\sqrt{\sigma^2+\delta}-\sigma\right)\left[R\frac{\partial u_i(h_\alpha)}{\partial E} + 2\frac{\partial u_i(h_\alpha)}{\partial\sigma^2}\right] + \frac{\partial u_i(h_\alpha)}{\partial\sigma^2}2\alpha\left(\sqrt{\sigma^2+\delta}-\sigma\right)^2.$$

Since $\|(E(h_\alpha), Var(h_\alpha)) - (E, \sigma^2)\| < \varepsilon$,

$-2\frac{\frac{\partial u_1(h_\alpha)}{\partial \sigma^2}}{\frac{\partial u_1(h_\alpha)}{\partial E}} = r_1(E(h_\alpha), Var(h_\alpha)) < R < r_2(E(h_\alpha), Var(h_\alpha)) = -2\frac{\frac{\partial u_2(h_\alpha)}{\partial \sigma^2}}{\frac{\partial u_2(h_\alpha)}{\partial E}}$.

So, $R\frac{\partial u_1(h_\alpha)}{\partial E} + 2\frac{\partial u_1(h_\alpha)}{\partial \sigma^2} > 0$; and since $\frac{\partial u}{\partial \sigma^2} \leq 0$, we have:

$\frac{\partial u_1(h_\alpha)}{\partial \alpha} \leq -\sqrt{\sigma^2+\delta}\left(\sqrt{\sigma^2+\delta} - \sigma\right)\left[R\frac{\partial u_i(h_\alpha)}{\partial E} + 2\frac{\partial u_i(h_\alpha)}{\partial \sigma^2}\right] < 0$ for all $\alpha \in [0,1]$.

Thus, agent 1 chooses $\alpha = 0$, and $h_2 \in C_1$.

But $R\frac{\partial u_2(h_\alpha)}{\partial E} + 2\frac{\partial u_2(h_\alpha)}{\partial \sigma^2} < 0$, and $\left.\frac{\partial u_2(h_\alpha)}{\partial \alpha}\right|_{\alpha=0} > 0$. Thus, agent 2 chooses $\alpha^* > 0$, which corresponds to h_{α^*} with mean and variance less than those of h_2.

To summarize, $h_2 \in C_1$, $h_{\alpha^*} \in C_2$, and $Var(h_2) > Var(h_{\alpha^*})$, but $h_2 \notin C_2$ and $h_{\alpha^*} \notin C_1$. This contradicts (**4***). ■

References

Epstein, L., 1985, Decreasing Risk Aversion and Mean-Variance Analysis, Econometrica 53, 945-961.

Kusuda, K., 2001, Measures of Variance Aversion and Decreasing Variance Aversion for Mean-Variance Utilities and Portfolio Choice Problems, mimeo, University of Minnesota.

Machina, M., 1982, "Expected Utility" Analysis Without the Independence Axiom, Econometrica 50, 277-323.

Pratt, J., 1964, Risk Aversion in the Small and the Large, Econometrica 32, 122-136.

Ross, S., 1981, Some Stronger Measures of Risk Aversion in the Small and the Large with Applications, Econometrica 49, 621-638.

Wang Z. and J. Werner, 1994, Portfolio Characterization of Risk Aversion, Economics Letters 45, 259-265.