# A Comment on "Sequential Equilibria"

**by**

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# A COMMENT ON "SEQUENTIAL EQUILIBRIA"

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## 1. Introduction

In addition to introducing the path-breaking concept of sequential equilibrium, Kreps and Wilson (1982) (henceforth KW) contains three insightful theorems which derive the geometry of the set of sequential equilibrium assessments, the finiteness of the set of sequential equilibrium outcomes, and the perfection of strict sequential equilibria. These derivations depend upon Lemmas A1 and A2 in the KW appendix.

Section 3 of this paper notes that the KW proofs of these lemmas contain a nontrivial fallacy. Section 4 repairs these proofs by means of Streufert (2006b).

## 2. Definitions

This section recalls the relevant KW terminology and defines an example which will be useful in Section 3.

This paragraph and Figure 2.1 define a game form $[T, \prec, A, \alpha, H, \rho]$. The set $T$ of nodes contains the set $X = \{o, oL, oR, oL\ell, oLr, oR\ell\}$ of decision nodes, which in turn contains the set $W = \{o\}$ of initial nodes. The set $W$ is given the trivial distribution $\rho = (\rho(o)) = (1)$, and the set $X$ is partitioned into the information sets $h \in H = \{\{o\}, \{oL, oR\}, \{oL\ell, oLr, oR\ell\}\}$. Let $H(x)$ denote the information set $h$ which contains $x$. Finally, let $A = \{L, R, \ell, r, \delta, \varepsilon\}$ be the set of actions $a$, let $A(h)$ be the set of actions available from information set $h$, and let $\alpha(x)$ be the last action taken to reach a non-initial node $x$.

A strategy profile is a function $\pi{:}A \to [0,1]$ such that $(\forall h)\ \Sigma_{a \in A(h)} \pi(a) = 1$. A belief system is a function $\mu{:}X \to [0,1]$ such that $(\forall h)\ \Sigma_{x \in h} \mu(x)$

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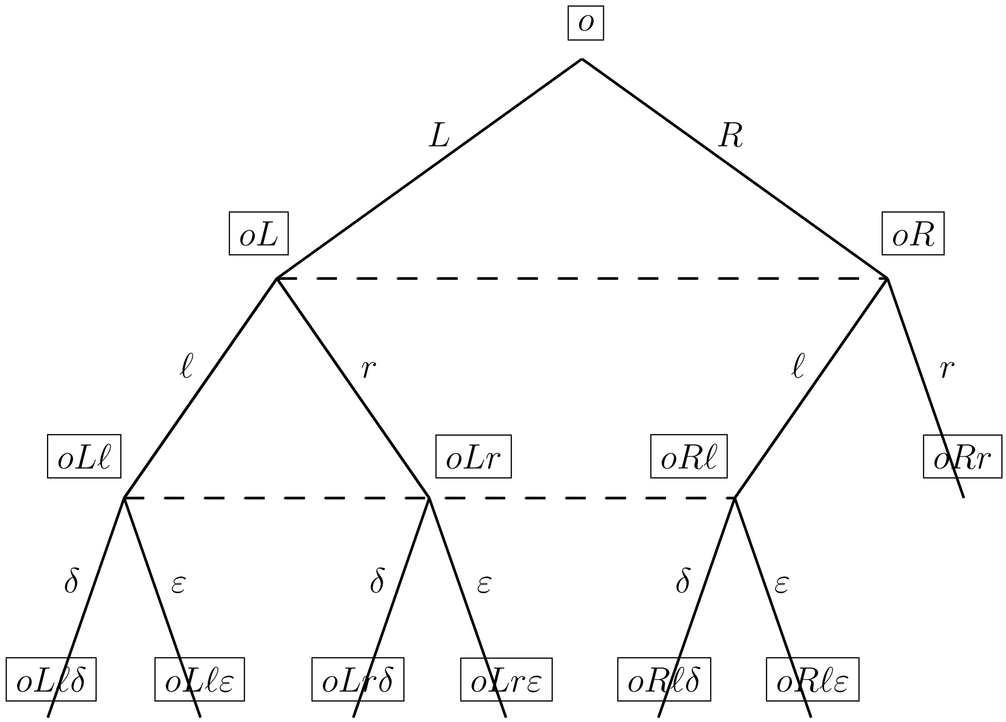


FIGURE 2.1

= 1. An assessment is a strategy-belief pair $(\pi, \mu)$. As on KW page 872, let $\Psi^0$ consist of those strictly positive assessments for which

$$(\forall x)\ \mu(x) = \frac{\rho \circ p_{\ell(x)}(x) \cdot \Pi_{k=0}^{\ell(x)-1} \pi \circ \alpha \circ p_k(x)}{\Sigma_{x' \in H(x)}\ \rho \circ p_{\ell(x')}(x') \cdot \Pi_{k=0}^{\ell(x')-1} \pi \circ \alpha \circ p_k(x')}\ ,$$

where $p_k(x)$ is the $k$th predecessor of node $x$, and $\ell(x)$ is the number of its predecessors. An assessment is *consistent* if it is the limit of a sequence $\{\pi_n, \mu_n\}$ in $\Psi^0$. For instance, in the example, the $(\pi, \mu)$ defined in the second lines of

(1)

| $a$ | $L$ | $R$ | $\ell$ | $r$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|---|---|---|
| $\pi_n(a)$ | $\frac{n^{-2}}{n^{-2}+1}$ | $\frac{1}{n^{-2}+1}$ | $\frac{n^{-1}}{n^{-1}+1}$ | $\frac{1}{n^{-1}+1}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $\pi(a)$ | 0 | 1 | 0 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |

and

| $x$ | $o$ | $oL$ | $oR$ | $oL\ell$ | $oLr$ | $oR\ell$ |
|---|---|---|---|---|---|---|
| $\mu_n(x)$ | 1 | $\frac{n^{-2}}{n^{-2}+1}$ | $\frac{1}{n^{-2}+1}$ | $\frac{n^{-3}}{n^{-3}+n^{-2}+n^{-1}}$ | $\frac{n^{-2}}{n^{-3}+n^{-2}+n^{-1}}$ | $\frac{n^{-1}}{n^{-3}+n^{-2}+n^{-1}}$ |
| $\mu(x)$ | 1 | 0 | 1 | 0 | 0 | 1 |

is consistent because the second line in each table is the limit of its first line, and because the $(\pi^n, \mu^n)$ defined in the first lines of the tables is within $\Psi^0$ for any value of $n$.

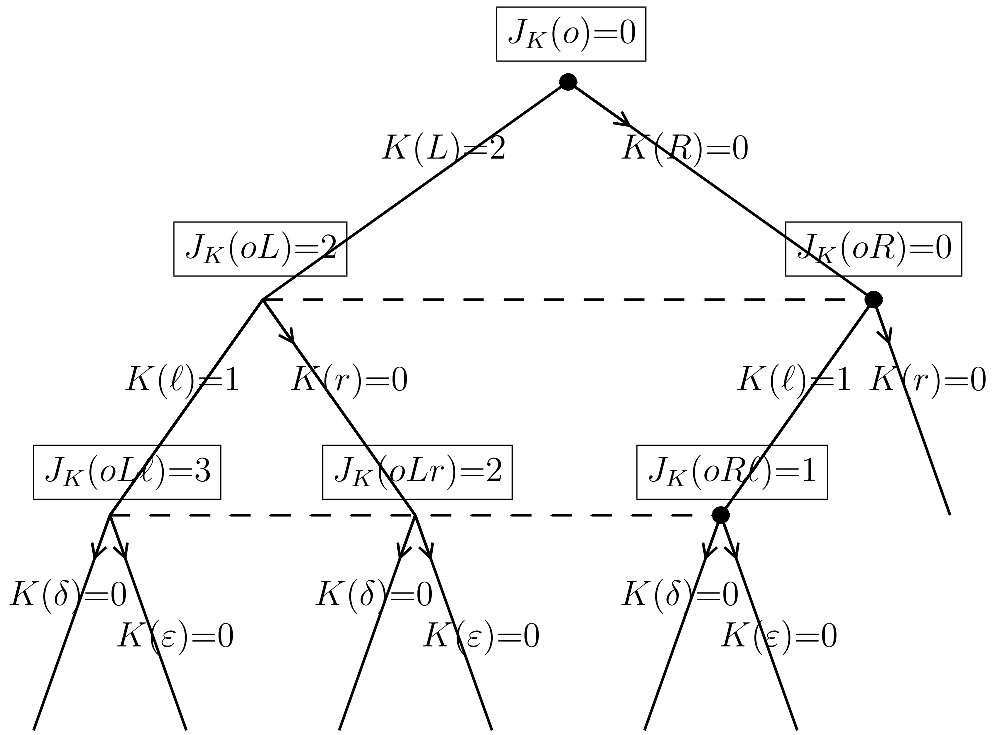


FIGURE 2.2

The following two paragraphs discuss two less familiar definitions. Both concern subsets of $X{\cup}A$. Note that KW page 880 calls *any* subset of $X{\cup}A$ a *basis*.

As on KW page 880, a basis $b$ is *consistent* if the set

$$\Psi_b = \{ \text{ consistent } (\pi,\mu) \mid (\forall a)\ a{\in}b \text{ iff } \pi(a){>}0 \text{ and } (\forall x)\ x{\in}b \text{ iff } \mu(x){>}0 \ \} \tag{2}$$

is nonempty. For instance, in the example, the basis

$$b = \{R, r, \delta, \varepsilon, o, oR, oR\ell\} \tag{3}$$

is consistent because the $(\pi,\mu)$ defined in (1) belongs to $\Psi_b$.

As on KW Page 887, a basis $b$ is *labelled* by a nonnegative-integer-valued function $K{:}A{\rightarrow}\mathbb{Z}_+$ if

$$(\forall h)(\exists a{\in}A(h))\ K(a) = 0 \tag{4a}$$

$$(\forall a)\ a \in b \text{ iff } K(a) = 0 \tag{4b}$$

$$(\forall x)\ x \in b \text{ iff } x \in \text{argmin}\{J_K(x')|x'{\in}H(x)\}\ , \tag{4c}$$

where $J_K{:}X{\rightarrow}\mathbb{Z}_+$ is defined by

$$J_K(x) = \Sigma_{k=0}^{\ell(x)-1} K{\circ}\alpha{\circ}p_k(x)\ . \tag{5}$$

For instance, in the example, the $b$ defined in (3) is labelled by the $K$ defined in Figure 2.2. To see this, first note that the figure calculates $J_K$, then note that the figure also depicts $b$ with arrows for actions and dots for nodes, and finally, inspect each of the three conditions in (4).

## 3. A Fallacy

On KW page 888, Lemma A2's proof draws upon Lemma A1. On KW page 887, Lemma A1 appears as follows.

*"Lemma A1: The basis $b$ is consistent ($\Psi_b$ is nonempty) if and only if a $b$ labelling exists."*

In other words, Lemma A1 states that a basis is consistent iff it can be labelled. (The consistency of $b$ is synonymous with the nonemptiness of $\Psi_b$.)

Lemma A1's proof appears on KW page 887. I find the argument unconvincing. In particular, its second paragraph does not show how to label an arbitrary consistent basis. The remainder of this section examines this paragraph sentence-by-sentence (the fallacy occurs in the very last sentence).

*"Now suppose that $b$ is a consistent basis."*

In Section 2's example, the $b$ defined at (3) is consistent. The proof should tell us how to label this basis $b$ with a function $K$.

*"Since $\Psi_b$ is nonempty, there exists a sequence $\{(\mu_n, \pi_n)\} \subseteq \Psi^\circ$ with the limit $(\mu, \pi)$ belonging to $\Psi_b$."*

In the example, the sequence $\{(\mu_n, \pi_n)\} \subseteq \Psi^0$ defined in the first lines of (1) has the limit $(\mu, \pi)$ defined in the second lines of (1) and this $(\mu, \pi)$ is an element of $\Psi_b$ for the $b$ defined at (3).

*"Let $M$ denote the finite set of all first degree, single term multinomials with coefficient one in the symbols $a \in A$."*

Thus $M$ consists of 1, each action, each pair of actions, each triple of actions, and so forth. In the example, elements of $M$ include 1, $R$, and $R\ell$, and accordingly, the elements of $M$ will be useful in describing the paths that reach nodes (1 describes the empty path taken to the initial node). (The set $M$ also happens to contain many other multinomials like $RL\ell r$ which do not correspond to paths that reach nodes, but these extra multinomials don't impose much of a burden.)

*“For $m \in M$, let $m_n$ represent $m$ evaluated with $a = \pi_n(a)$.”*

In the example, if $m = R\ell$, then the number $m_n = (R\ell)_n$ is $R\ell$ evaluated with $R = \pi_n(R)$ and $\ell = \pi_n(\ell)$, which reduces to $\pi_n(R)\pi_n(\ell)$, which by the definition (1) of $\{\pi_n\}_n$ is $\frac{1}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$.

*“Without loss of generality, we can assume that for every pair $m$ and $m'$ from $M$, the sequence $m_n/m'_n$ converges either to zero, to infinity, or to some strictly positive number. (This is wlog because we can look along a subsequence of $\{(\mu_n, \pi_n)\}$ for which it is true.)”*

In the example, consider $m = R\ell$ and $m' = L\ell$. Recall from the last step that $(R\ell)_n$ is $\frac{1}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$. Similarly, $(L\ell)_n$ is $\frac{n^{-2}}{n^{-2}+1}\frac{n^{-1}}{n^{-1}+1}$. Thus $(R\ell)_n/(L\ell)_n$ is $1/n^{-2} = n^2$, which happens to converge to infinity. In fact, all such ratio sequences in the example converge either to zero, to infinity, or to some strictly positive number (in other words, the subsequence argument is unnecessary in the example).

*“Define $m \dot{<} m'$ if $lim_n m_n/m'_n = \infty$; then $\dot{<}$ is an asymmetric and negatively transitive binary relation on $M$.”*

In the example, $R\ell \dot{<} L\ell$ because $\lim_n (R\ell)_n/(L\ell)_n = \infty$ by the last step. Many similar calculations reveal that the restriction of $\dot{<}$ to $\{1, L, R, L\ell, Lr, R\ell\}^2$ is the set of pairs $(m, m')$ that receive a dot • in the table

(6)

| $J(m')$ | $m'$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $R\ell$ | • | | • | | | | |
| 2 | $Lr$ | • | | • | | | • | |
| 4 | $L\ell$ | • | • | • | | • | • | |
| 0 | $R$ | | | | | | | |
| 2 | $L$ | • | | • | | | • | |
| 0 | 1 | | | | | | | |
| | | 1 | $L$ | $R$ | $L\ell$ | $Lr$ | $R\ell$ | $m$ |
| | | 0 | 2 | 0 | 4 | 2 | 1 | $J(m)$ |

($R\ell \dot{<} L\ell$ appears as the dot with $R\ell$ on the horizontal axis and $L\ell$ on the vertical axis). (The binary relation $\dot{<}$ is a much larger subset of $M^2$, but this is unimportant because the elements of $M$ outside of $\{1, L, R, L\ell, Lr, r\ell\}$ do not correspond to paths that reach decision nodes.)

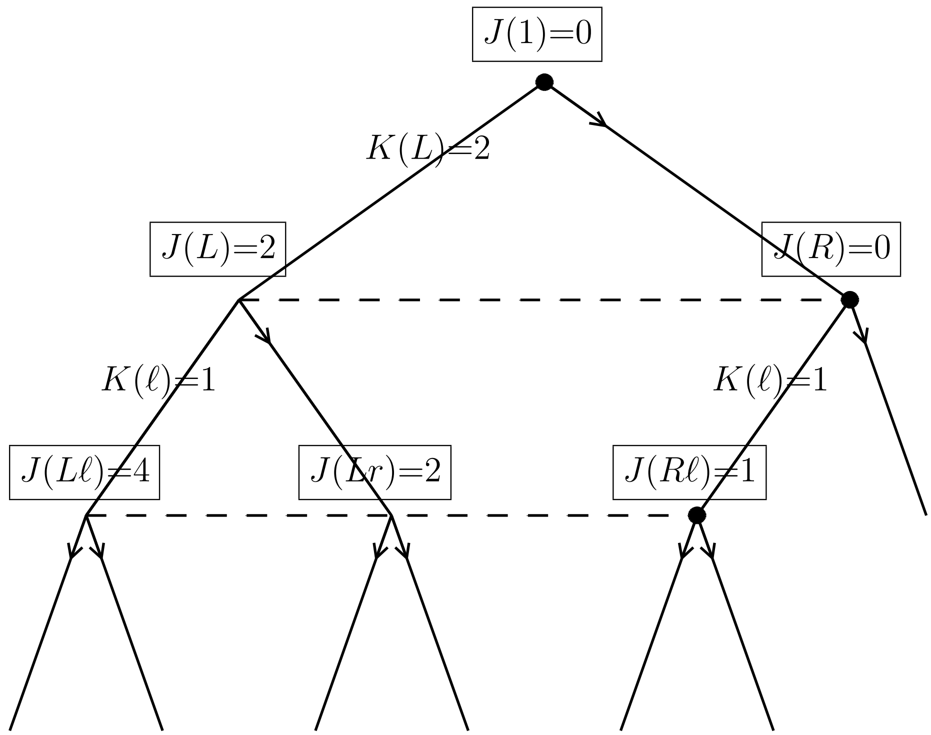


FIGURE 3.1

*"Since $M$ is finite there exists an integer valued function $J$ on $M$ with $m \dot{<} m'$ if and only if $J(m) < J(m')$. We can pick $J$ so that $J(m) = 0$ for the $\dot{<}$−least $m$—then $J(m) \geq 0$ for all $m$."*

In the example, such a $J$ appears in the last row (and first column) of (6). The same $J$ also appears in the boxes of Figure 3.1. (To be precise, we are only concerning ourselves with the restriction of $J$ to $\{1, L, R, L\ell, Lr, R\ell\}$.)

*"For each $x \in X$ there is an associated $m^x \in M$, namely $m^x = \Pi_{\ell=0}^{\ell(x)-1} \alpha(p_\ell(x))$." (For $x \in W$, $m^x = 1$.)*

Each $m^x$ is the list of actions leading to $x$. In the example, $m^{oR\ell} = R\ell$ and $\{m^x | x{\in}X\} = \{1, L, R, L\ell, Lr, R\ell\}$.

*"Now for each $a$ pick an arbitrary $x \in H(a)$ such that $J(m^x)$ is minimal over $x \in H(a)$ and define*

$$K(a) = J(m^x{\cdot}a) - J(m^x)\ ." \tag{7}$$

(First an insignificant remark: I take $H(a)$ to be the information set from which the action $a$ can be chosen. In other words, I take $H(a)$ to equal $A^{-1}(a)$, as defined on KW Page 867.)

Consider the action $a = \ell$ in the example. It can be chosen from the information set $H(\ell) = \{oL, oR\}$, and from the bottom row in (6) or from the boxes in Figure 3.1, we have that $J(L) = 2$ and $J(R) = 0$. Hence $x = oR$ is, in the above words from KW evaluated at $a = \ell$, "an arbitrary $x \in H(\ell)$ such that $J(m^x)$ is minimal over $x \in H(\ell)$" (in fact, it is the only such $x$). Hence (7) sets

$$K(\ell) = J(m^{oR}{\cdot}\ell) - J(m^{oR}) = J(R\ell) - J(R) = 1 \ . \tag{8}$$

Similarly consider the action $a = L$ in the example. It can be chosen from the information set $H(L) = \{o\}$. Hence $x = o$ is (trivially) "an arbitrary $x \in H(L)$ such that $J(m^x)$ is minimal over $x \in H(L)$." Thus (7) sets

$$K(L) = J(m^o{\cdot}L) - J(m^o) = J(L) - J(1) = 2 \ . \tag{9}$$

*"We leave to the reader the relatively easy tasks of proving that $K(a)$ is well-defined (i.e., the choice of a $J(m^x)$-minimal $x \in H(a)$ is irrelevant) and that $K$ so defined is a b labelling (with, of course, $J_K(x) = J(m^x)$)."*

The equation $J_K(x) = J(m^x)$ cannot be derived. Consider the example. There $J_K(oL\ell) = K(L)+K(\ell) = 2+1 = 3$ by (5), (8), and (9). Yet $J(m^{oL\ell}) = J(L\ell) = 4$ by the definition of $J$ in the last row of (6).

The difficulty lies in the choice of the function $J$. I deliberately chose $J(L\ell) = 4$. Had I alternatively chosen $J(L\ell) = 3$, there would have been no problem with this example at the last stage of the proof.

However, making a judicious choice of $J$ is a nontrivial problem. Not any representation of the binary relation $\dot{<}$ will do. Rather, it has to be additive across actions. Finding that additive representation lies at the heart of Streufert (2006a) and Streufert (2006b). A theorem from the second of these papers is employed in the next section to prove the KW lemmas.

## 4. Proofs of KW Lemmas A1 and A2

### 4.1. Preliminary Remarks

For any nonpositive-integer-valued function $\varepsilon{:}A{\to}\mathbb{Z}_-$ and any node $x$, define the set

$$H^\varepsilon(x) = \text{argmax}\{\ \Sigma_{k=0}^{\ell(x')-1} \varepsilon{\circ}\alpha{\circ}p_k(x') \mid x'{\in}H(x)\ \} \ .$$

Corollary 4.1 is equivalent to Streufert (2006b, Corollary 2.2). Note that the $\varepsilon$ here differs from the $e$ in Streufert (2006b) to the extent that $\varepsilon$ cannot assume positive values.

COROLLARY 4.1. *In any game form* $[T,\prec,A,\alpha,\rho,H]$, *an assessment* $(\pi,\mu)$ *is consistent iff there exists* $\xi{:}A{\to}(0,\infty)$ *and* $\varepsilon{:}A{\to}\mathbb{Z}_-$ *such that*

$$(\forall a)\ \pi(a) = \begin{pmatrix} \xi(a) & \textit{if } \varepsilon(a)=0 \\ 0 & \textit{if } \varepsilon(a)<0 \end{pmatrix} \ \textit{and} \tag{10a}$$

$$(\forall x)\ \mu(x) = \begin{pmatrix} \dfrac{\rho{\circ}p_{\ell(x)}(x){\cdot}\Pi_{k=0}^{\ell(x)-1}\xi{\circ}\alpha{\circ}p_k(x)}{\Sigma_{x'\in H^\varepsilon(x)}\rho{\circ}p_{\ell(x')}{\cdot}\Pi_{k=0}^{\ell(x')-1}\xi{\circ}\alpha{\circ}p_k(x')} & \textit{if } x{\in}H^\varepsilon(x) \\ 0 & \textit{if } x{\notin}H^\varepsilon(x) \end{pmatrix} . \tag{10b}$$

*Proof.* Corollary 4.1 and Streufert (2006b, Corollary 2.2) are equivalent after $(\xi,\varepsilon)$ has been identified with $(c,e)$. Here the nonnegativity in $\varepsilon$ is implicit, there the nonnegativity of $e$ appears explicitly just before the corollary's two equations. □

The next three lemmas are very simple, but they will each be employed several times in future proofs. To get oriented, note that any two of (11), (12), and (14) imply the third.

LEMMA 4.2. *If* $(\pi,\mu)$ *and* $(\xi,\varepsilon)$ *satisfy (10), then*

$$(\forall a)\ \pi(a)>0 \ \textit{iff}\ \varepsilon(a)=0 \ \textit{and} \tag{11a}$$

$$(\forall x)\ \mu(x)>0 \ \textit{iff}\ x\in H^\varepsilon(x)\ . \tag{11b}$$

*Proof.* Obvious. □

LEMMA 4.3. $(\pi,\mu)\in\Psi_b$ *iff* $(\pi,\mu)$ *is consistent,*

$$(\forall a)\ a\in b \ \textit{iff}\ \pi(x)>0\ ,\ \textit{and} \tag{12a}$$

$$(\forall x)\ x\in b \ \textit{iff}\ \mu(x)>0\ . \tag{12b}$$

*Proof.* This follows from the definition (2) of $\Psi_b$. □

LEMMA 4.4. *A basis* $b$ *can be labelled iff there is an* $\varepsilon{:}A{\to}\mathbb{Z}_-$ *such that*

$$(\forall h)(\exists a{\in}A(h))\ \varepsilon(a)=0\ , \tag{13}$$

$$(\forall a)\ a\in b \ \textit{iff}\ \varepsilon(a)=0\ ,\ \textit{and} \tag{14a}$$

$$(\forall x)\ x\in b \ \textit{iff}\ x\in H^\varepsilon(x)\ . \tag{14b}$$

*Proof.* (4) is equivalent to the combination of (13) and (14) after $J_K$ has been substituted out, and $K$ and $-\varepsilon$ have been identified. □

## 4.2. Lemma A1

Lemma 4.5 (KW Lemma A1). *A basis is consistent iff it can be labelled.*

*Proof.* Suppose $b$ is consistent. Then $\Psi_b \neq \varnothing$, and thus by Lemma 4.3 there is a consistent assessment $(\pi, \mu)$ satisfying (12). Because $(\pi, \mu)$ is consistent, Corollary 4.1 yields $(\xi, \varepsilon)$ satisfying (10), which by Lemma 4.2 yields (11). (12) and (11) together yield (14). (10a) and the well-definition of $\pi$ yield (13). Hence, by Lemma 4.4, $b$ can be labelled.

Conversely, suppose that $b$ can be labelled. Then by Lemma 4.4 there exists some $\varepsilon$ which satisfies (13) and (14). Then define $\xi$ by

$$\xi(a) = 1/|\{ \ a'{\in}A(H(a)) \mid \varepsilon(a'){=}0 \ \}| \ .$$

Because of (13) and the normalization in the definition of $\xi$, we can construct $\pi$ and $\mu$ to satisfy (10). Then by Corollary 4.1, $(\pi, \mu)$ is consistent. Further, (10) yields (11) by Lemma 4.2, and then, (14) and (11) yield (12). Hence, by Lemma 4.3, $(\pi, \mu) \in \Psi_b$. Hence $b$ is consistent. □

## 4.3. Lemma A2

As on KW page 888, define

$$\Xi_b = \{ \ \xi{:}A{\rightarrow}(0,\infty) \mid (\forall h) \ \Sigma_{a \in b \cap A(h)} \xi(a) = 1 \ \} \ ,$$

let $\pi^b$ map $\xi \in \Xi_b$ to

(15a)
$$\pi^b(\xi)(a) = \begin{pmatrix} \xi(a) & \text{if } a \in b \\ 0 & \text{if } a \notin b \end{pmatrix} \ ,$$

and let $\mu^b$ map $\xi \in \Xi_b$ to

(15b)
$$\mu^b(\xi)(x) = \begin{pmatrix} \dfrac{\rho{\circ}p_{\ell(x)}(x){\cdot}\Pi_{k=0}^{\ell(x)-1}\xi{\circ}\alpha{\circ}p_k(x)}{\Sigma_{x' \in b \cap H(x)}\rho{\circ}p_{\ell(x')}{\cdot}\Pi_{k=0}^{\ell(x')-1}\xi{\circ}\alpha{\circ}p_k(x')} & \text{if } x \in b \\ 0 & \text{if } x \notin b \end{pmatrix}$$

(the KW multinomials $m^x$ have been substituted out, and the restriction $(\forall w)\ \rho(w) = 1/|W|$ arbitrarily imposed at the start of KW Section A.1 has been relaxed).

Lemma 4.6 (KW Lemma A2). *For any consistent $b$, $\Psi_b$ is the image of $\Xi_b$ under the mapping $(\pi^b, \mu^b)$.*

*Proof.* Take any assessment $(\pi,\mu)$ in $\Psi_b$. Thus by Lemma 4.3, we have (12). Further, since $(\pi,\mu)$ is consistent, Corollary 4.1 yields the existence of $(\xi,\varepsilon)$ which satisfy (10), which by Lemma 4.2 yields (11). Then (12) and (11) imply (14). We now assemble three facts. [a] $\pi = \pi^b(\xi)$ by (10a), (14a), and definition (15a). [b] $\mu = \mu^b(\xi)$ by (10b), (14b), and definition (15b), and by the fact that $(\forall x)\ b \cap H(x) = H^\varepsilon(x)$ by (14b). [c] $\xi \in \Xi_b$ by (10a) and the well-definition of $\pi$. By these three facts, $(\pi,\mu)$ is in the image of $\Xi_b$ under $(\pi^b,\mu^b)$.

Conversely, take any consistent $b$ and any $\xi \in \Xi_b$. By Lemmas 4.5 and 4.4, there is some $\varepsilon$ which satisfies (14). Then, $\pi^b(\xi)$ satisfies (10a) by (15a) and (14a). Further, (14b) yields $(\forall x)\ b \cap H(x) = H^\varepsilon(x)$, and thus, $\mu^b(\xi)$ satisfies (10b) by (15b) and (14b). Since $(\pi^b(\xi),\mu^b(\xi))$ satisfies (10) by the last two sentences, Corollary 4.1 yields that $(\pi^b(\xi),\mu^b(\xi))$ is consistent. Therefore, since (15) yields that $(\pi^b(\xi),\mu^b(\xi))$ satisfies (12), Lemma 4.3 yields $(\pi^b(\xi),\mu^b(\xi)) \in \Psi_b$. □

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